Solution of Integral Equation using Second and Third Order B-Spline Wavelets

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Abstract
It was proven that semi-orthogonal wavelets approximate the solution of integral equation very finely over the orthogonal wavelets. Here we used the compactly supported semi-orthogonal B-spline wavelets generated in our paper "Compactly Supported B-spline Wavelets with Orthonormal Scaling Functions" satisfying the Daubechies conditions, to solve the Fredholm integral equation. The generated wavelets satisfies all the properties on the bounded interval. The method is computationally easy, which is illustrated with two examples whose solution closely resembles the exact solution as the order of wavelet increases.

Index terms—B-spline wavelets, dual wavelets, integral equation.

1 i. Introduction
Integral equations are find very vast usage in many areas of engineering, physics, applied mathematics and many more. Here we seek to resolve a class of integral equation called Fredholm integral equation. There are various methods like variational method, collocation type method and integrated collocation method are known to estimate the solution of integral [1]. Some of the methods convert the integral equation into non linear equation while in some other method it transform to a set of algebraic equations.

Wavelets due to its outstanding properties like vanishing moment, compact support, are good candidates for providing fast algorithm in numerical aspects in approximating [3,4,5,6]. In the present paper, we apply compactly supported semi orthogonal B-Spline wavelet generated in our paper [7] for bounded interval to solve the linear Fredholm integral equation of form \( \int_a^b f(x)\phi(x)dx = \int_a^b k(x)\phi(x)dx \) where \( f \) and \( k \) are given continuous function. Due to the interesting features like smoothness which increases with order of vanishing moment and closed form expression of compactly supported spline wavelets, it was widely used in solving numerical problems. The wavelet formed satisfy all the properties on a bounded interval. In [8] it was shown that semi-orthogonal wavelets are better than orthogonal for integral equation application.

ii. Second and Third Order b-spline Wavelets on \([0,1]\)
The wavelets are generally defined as \( \psi^n(x) = \psi^n(2x, n) \) to give a complete wavelet in interval \([0,1]\). Here the actual coordinate position \( x \) is related to \( s \) as \( x = \frac{s}{2^n} \). The second order B-spline wavelet obtained in [7] are given by

\[
\psi^{2,0}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x \geq 1 
\end{cases}
\]

\[
\psi^{2,1}(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 - 2x & \text{if } 1 \leq x < 2 \\ 0 & \text{if } x \geq 2 
\end{cases}
\]

\[
\psi^{2,2}(x) = \begin{cases} x^2 - 2x + 1 & \text{if } x \leq 1 \\ 0 & \text{if } x > 1 
\end{cases}
\]

\[
\psi^{2,3}(x) = \begin{cases} 2x^3 - 3x^2 + 1 & \text{if } x \leq 1 \\ 0 & \text{if } x > 1 
\end{cases}
\]

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\[
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\end{cases}
\]

\[
\psi^{2,3}(x) = \begin{cases} 2x^3 - 3x^2 + 1 & \text{if } x \leq 1 \\ 0 & \text{if } x > 1 
\end{cases}
\]
The inner wavelet functions are obtained as:
\[ \phi_{-2,0}(t) = \frac{1}{\sqrt{8}} \left( 4t^2 - 7 - 2t^3 + 2t \right) \]
\[ \psi_{-2,0}(t) = \frac{1}{\sqrt{8}} \left( -3t^2 + 10 + 3t - 2t^3 \right) \]
\[ \phi_{-2,-1}(t) = \frac{1}{\sqrt{8}} \left( -6t^2 + 11 + 3t^2 - 3t \right) \]
\[ \psi_{-2,-1}(t) = \frac{1}{\sqrt{8}} \left( -4t^2 + 10 + 3t - 3t^3 \right) \]
\[ \phi_{-2,-2}(t) = \frac{1}{\sqrt{8}} \left( 4t^2 - 7 - 2t^3 + 2t \right) \]
\[ \psi_{-2,-2}(t) = \frac{1}{\sqrt{8}} \left( -3t^2 + 10 + 3t - 2t^3 \right) \]

The L.H.S and R.H.S boundary wavelet function are given as:
\[ \phi_{1,0}(t) = \frac{1}{\sqrt{8}} \left( 4t^2 - 7 - 2t^3 + 2t \right) \]
\[ \psi_{1,0}(t) = \frac{1}{\sqrt{8}} \left( -3t^2 + 10 + 3t - 2t^3 \right) \]

The third order B-spline scaling function and B-spline wavelet function are given by:
\[ \phi_{3,0}(t) = \frac{1}{\sqrt{8}} \left( 4t^2 - 7 - 2t^3 + 2t \right) \]
\[ \psi_{3,0}(t) = \frac{1}{\sqrt{8}} \left( -3t^2 + 10 + 3t - 2t^3 \right) \]

The L.H.S and R.H.S boundary scaling function and wavelet function are given by:
\[ \phi_{3,-1}(t) = \frac{1}{\sqrt{8}} \left( 4t^2 - 7 - 2t^3 + 2t \right) \]
\[ \psi_{3,-1}(t) = \frac{1}{\sqrt{8}} \left( -3t^2 + 10 + 3t - 2t^3 \right) \]

The third order B-spline scaling function and B-spline wavelet function are given by:
\[ \phi_{3,0}(t) = \frac{1}{\sqrt{8}} \left( 4t^2 - 7 - 2t^3 + 2t \right) \]
\[ \psi_{3,0}(t) = \frac{1}{\sqrt{8}} \left( -3t^2 + 10 + 3t - 2t^3 \right) \]

The inner third order B-spline scaling function are obtained by substituting \( t = 3 \) and \( t = 0,1,2,3,4,5 \) in eqn(2.3). And the inner wavelet functions are obtained by putting \( t = 3 \) and \( t = 0,1,2,3,4,5 \) in equation (2.4).

Fig 2.1 shows the B-spline wavelets for \( t = 2 \) and \( t = 3 \). where

The L.H.S and R.H.S boundary scaling function and wavelet function are given by

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The inner wavelet functions are obtained as:

\[ \phi_{3,0}(t) = \frac{1}{\sqrt{8}} \left( 4t^2 - 7 - 2t^3 + 2t \right) \]
\[ \psi_{3,0}(t) = \frac{1}{\sqrt{8}} \left( -3t^2 + 10 + 3t - 2t^3 \right) \]

The inner third order B-spline scaling function are obtained by substituting \( t = 3 \) and \( t = 0,1,2,3,4,5 \) in eqn(2.3). And the inner wavelet functions are obtained by putting \( t = 3 \) and \( t = 0,1,2,3,4,5 \) in equation (2.4).

Fig 2.1 shows the B-spline wavelets for \( t = 2 \) and \( t = 3 \). where

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where \( \delta \otimes \delta \) and \( \delta \otimes \delta \) are convolution products. The convolution products of second order B-spline wavelet can be defined as: 

\[
\delta \otimes \delta(\tau) = \sum_{k=-\infty}^{\infty} \delta(k) \cdot \delta(\tau-k)
\]

Using these convolution products, we can solve the integral equation (4.1). The solutions obtained for \( m=2 \) and \( m=3 \) are given in Table 5.2. In this way, from equation (4.2), we get the numerical solution of the integral equation (4.1). Similar process is applied for higher order wavelets and higher octave levels.

### Conclusion

In this paper, we present a method to solve linear Fredholm integral equations. Our approximation is based on compactly supported semi-orthogonal B-spline wavelet generated in our previous paper. Two examples are illustrated to check the validity and significance of the proposed technique. The solution of the examples reveals that the exactness of solution increases as the order of B-spline wavelet and octave level increases.

Figure 1: 2.1
iv. Fredholm Integral Equations
Here we consider Fredholm Integral Equations of type
\[ ??(??) = \delta ??" ??(??) + ? ??(??, ??) 1 0 \]

and solve this equation by second order B-spline wavelets for ?? = 2. Let first approximate \( y(x) \) as
\[ ??(??) = ?? ?? ?? P = ?? \delta ??" ?? + ??? ?? ?? \]
where ??(??) = ?? 2,-1 , ?? 2,0 , ? , ?? 2,3 , \delta ??" ?? 2,-1 , . . , ?? \delta ??" ?? 2,2 ?

Also denoted as ??(??) = [\delta ??" ?? 1 , \delta ??" ?? 2 , ? , \delta ??" ?? 6 , ? , \delta ??" ?? 9 ] ??

\[
\begin{pmatrix}
0.1347 & -0.034 & 0.0000 & 0.0000 \\
?? 2 = ? & -0.034 & 0.1667 & -0.0417 & 0.0000 & -0.0417 & 0.1667 & -0.049 ? \\
0.0000 & & & & & & 0.0000 & \\
\end{pmatrix}
\]

thus,

\[
\begin{pmatrix}
0.2819 \\
\end{pmatrix}
\]

\[ ?? ? ? ? T (x) \]

\[
\begin{pmatrix}
0.1347 & -0.034 & 0.0000 & 0.0000 \\
?? 2 = ? & -0.034 & 0.1667 & -0.0417 & 0.0000 & -0.0417 & 0.1667 & -0.049 ? \\
0.0000 & & & & & & 0.0000 & \\
\end{pmatrix}
\]

\[ ?? ?? ?? P = ?? \delta ??" ?? + ??? ?? ?? \]

\[
\begin{pmatrix}
0.2819 \\
\end{pmatrix}
\]

\[ ?? ? ? ? T (x) \]

\[
\begin{pmatrix}
0.1347 & -0.034 & 0.0000 & 0.0000 \\
?? 2 = ? & -0.034 & 0.1667 & -0.0417 & 0.0000 & -0.0417 & 0.1667 & -0.049 ? \\
0.0000 & & & & & & 0.0000 & \\
\end{pmatrix}
\]

\[ ?? ? ? ? T (x) \]

\[
\begin{pmatrix}
0.2819 \\
\end{pmatrix}
\]

\[ ?? ? ? ? T (x) \]

\[
\begin{pmatrix}
0.1347 & -0.034 & 0.0000 & 0.0000 \\
?? 2 = ? & -0.034 & 0.1667 & -0.0417 & 0.0000 & -0.0417 & 0.1667 & -0.049 ? \\
0.0000 & & & & & & 0.0000 & \\
\end{pmatrix}
\]

\[ ?? ? ? ? T (x) \]
Table 5.2

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Figure 3: Table 5.2

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Figure 4: Solution of Integral Equation using Second and Third Order B-Spline Wavelets

Table 5.2

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Figure 5: Table 5.2


