

Multi-Dimensional Rajan Transform

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Received: 14 December 2019 Accepted: 3 January 2020 Published: 15 January 2020

Abstract

In this paper, we describe the formulation of a novel transform called Multi-Dimensional Rajan Transform, which is an extension of Rajan Transform. Basically, Rajan Transform operates on a number sequence, whose length is a power of two. It transforms any sequence of arbitrary numbers into a sequence of interrelated numbers. As regards 2D Rajan Transform, there are two methods to implement it: (i) Row- Column method and (ii) Column-Row method. The 2D Rajan Transform obtained using the first method need not be the same as that obtained using second method. Similarly, one can implement 3-D Rajan Transform using the following approaches: (i) Row-Column-Depth approach, (ii) Row-Depth- Column approach, (iii) Column-Row-Depth approach, (iv) Column-Depth-Row approach, (v) Depth-Row-Column approach and (vi) Depth- Column-Row approach. This paper explains these approaches to implement two and three dimensional Rajan Transforms.

Index terms— discrete transforms, rajan transform, permutation invariant systems, homomorphic transforms.

Multi-Dimensional Rajan Transform K. Thiagarajan¹, Manish Prateek² & Ethirajan Govindarajan³
 $g(i) = x(i) + x(i+N/2)$; $0 \leq i < N/2$; $0 \leq j < N/2$ $h(j) = |x(i) - x(i-N/2)|$; $0 \leq j < N/2$; $0 \leq i < N/2$ Now each $(N/2)$ point segment is further divided into two halves each consisting of $N/4$ points so that the following holds good. $g1(k) = g(i) + g(j + (N/4))$; $0 \leq k < (N/4)$; $0 \leq j < (N/4)$ $g2(k) = |g(i) - g(i - (N/4))|$; $0 \leq k < (N/4)$; $0 \leq i < (N/4)$ $h1(k) = h(j) + h(j + (N/4))$; $0 \leq k < (N/4)$; $0 \leq j < (N/4)$ $h2(k) = |h(j) - h(j - (N/4))|$; $0 \leq k < (N/4)$; $0 \leq j < (N/4)$ This process is continued until no more division is possible. The total number of stages thus turns out to be $\log_2 N$. Let us denote the sum and difference operators respectively as '+' and '~'. Then the signal flow graph for the Rajan Transform of a number sequence of length 16 would be of the form shown in Fig. 1. If $x(n)$ is a number sequence of length $N = 2^k$; $k > 0$, then its Rajan Transform (RT) is denoted as $X(k)$. RT is applicable to any number sequence and it induces an isomorphism in a class of sequence, that is, it maps a domain set consisting of the cyclic and dyadic permutations of a sequence on to a range set consisting of sequence of the form $X(k)E(r)$ where $X(k)$ denotes the permutation invariant RT and $E(r)$ an encryption code corresponding to an element in the domain set. This map is a one-to-one and on to correspondence and an inverse map also exists. Hence, it is viewed as a transform. Fig. 1 shows a functional block diagram of a 16-point Rajan Transform algorithm.

It is to be noted that the map $x(n) \rightarrow X(k)E(r)$ is an isomorphic map, and the map $x(n) \rightarrow X(k)$ is a homomorphic map. The inverse function is called Inverse Rajan Transform and is applicable to cases where Rajan Transform is considered as an isomorphic function. Homomorphic functions do not have inverse functions. The operation of Inverse Rajan Transform is not explained here in this paper for lack of space and for its irrelevance in the formulation of multi-dimensional Rajan Transforms. Apart from multi-dimensional Rajan Transforms, which are conceptual extensions of onedimensional Rajan Transform, one can as well develop notions like "Set-Theoretic Rajan Transform" which are higher order algebraic tools that work on sequences of sets, functions and relations. These formulations are not in the scope of this paper and thus they are not presented.

1 Introduction

ajan transform essentially operates on a number sequence, whose length is a power of two. It transforms any sequence of arbitrary numbers into a sequence of interrelated numbers. The resulting sequence is called 'Rajan Spectrum'. More precisely, Rajan Transform is a homomorphic map that yields the same Rajan Spectrum for

46 an input sequence and all of its permuted versions. The definition and various outcomes of One-Dimensional
 47 Rajan Transform (1D-RT) are extended to multi-dimensional case. Onedimensional Rajan Transform is briefly
 48 explained on a need to have basis.

49 2 Definition of Rajan Transform

50 Rajan Transform is essentially a fast algorithm developed on the lines of Decimation-In-Frequency Fast Fourier
 51 Transform algorithms, but it is functionally different from the DIF-FFT algorithm. Given a number sequence
 52 $x(n)$ of length N , which is a power of 2, first it is divided into the first half and the second half each consisting
 53 of $(N/2)$ points so that the following holds good. As outlined earlier, one can implement twodimensional Rajan
 54 Transform in two different ways: (i) Row-Column method and (ii) Column-Row method. As stated previously,
 55 2D Rajan Spectrum obtained using first method need not be the same as the spectrum obtained using second
 56 method. This could be verified with the help of an example. Consider a two dimensional array $A = [x(m,n)]$.
 57 $0 \ 0 \ 0 \ 0 \ A = [x(m,n)] = 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0$ 2D-RT obtained using Row-Colum method

58 The RT spectra of the four rows of the array A are the four rows given in the array $[X_r(g,h)]$
 59 $0 \ 0 \ 0 \ 0 \ [X_r(g,h)] = 3 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1$

60 Next, the RT spectra of the four columns in the array $[X_r(g,h)]$ are given in the columns of the array $[X_{r,c}(k_1, k_2)]$
 61 (k_1, k_2) , which is the 2D-RT of the given array A .
 62 $5 \ 3 \ 3 \ 3 \ [X_{r,c}(k_1, k_2)] = 3 \ 1 \ 1 \ 1 \ 3 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1$ 2D-RT
 62 obtained using Column-Row method

63 The RT spectra of the four columns of the array A are the four columns given in the array $[X_c(g,h)]$
 64 $1 \ 3 \ 1 \ 0 \ [X_c(g,h)] = 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0$

65 Next, the RT spectra of the four rows in the array $[X_c(g,h)]$ are given in the rows of the array $[X_{c,r}(k_1, k_2)]$
 66 (k_1, k_2) , which is the 2D-RT of the given array A .
 67 $5 \ 1 \ 3 \ [X_{c,r}(k_1, k_2)] = 3 \ 1 \ 1 \ 3 \ 1 \ 1 \ 3 \ 1 \ 1$

68 With reference to the example presented above, it is proved that $[X_{r,c}(k_1, k_2)] = [X_{c,r}(k_1, k_2)]$ in this
 69 case. However, one can verify that $[X_{r,c}(K_1, k_2)] = [X_{c,r}(k_1, k_2)]$ for symmetric arrays. In order to do
 69 this, let us consider Hilbert matrix H_4 and obtain 2D-RT spectra using both methods.

70 $1 \ 2 \ 3 \ 4 \ H_4 = [x(m,n)] =$
 71 $2 \ 3 \ 4 \ 5 \ 3 \ 4 \ 5 \ 6 \ 4 \ 5 \ 6 \ 7$

72 The 2D-RT spectra of H_4 obtained using Row-Column method is $X_{r,c}(k_1, k_2)$ and it is shown below.
 73 $08 \ 16 \ [X_{r,c}(k_1, k_2)] =$

74 $08 \ 00 \ 00 \ 16 \ 00 \ 00 \ 00 \ 00$

75 The 2D-RT spectra of H_4 obtained using Column-Row method is $X_{c,r}(k_1, k_2)$ and it is shown below.

76 3 08 16

77 $[X_{c,r}(k_1, k_2)] = 08 \ 00 \ 00 \ 16 \ 00 \ 00 \ 00 \ 00$ In this case of symmetric matrix $[X_{r,c}(k_1, k_2)] = [X_{c,r}(K_1, k_2)]$.
 78 $2)$.

79 4 2D-RT Translation invariance property

80 Consider the two-dimensional array A showing aT like pattern.
 81 $0 \ 0 \ 0 \ A = [x(m,n)] = 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0$ B is the
 81 translated version of A .
 82 $0 \ 1 \ 1 \ B = [x_t(m,n)] = 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$

82 One can verify that $[X_{c,r}(K_1, k_2)]$ for both arrays A and B remain the same. Similarly, one can verify that
 83 $[X_{r,c}(k_1, k_2)]$ remain the same for both arrays A and B . This amounts to saying that 2D-Rajan Transform
 84 is essentially a translation invariant function, which could be effectively used in pattern recognition.

85 5 III.

86 6 Three-Dimensional Rajan Transform

87 The definition and various properties associated with two dimensional RT is extended to the threedimensional
 88 case also. Especially, the basic methods for implementing 2D Rajan Transform like Row-Column method and
 89 Column-Row method could further be generalized to implement 3D Rajan Transform. One can implement 3-D
 90 Rajan Transform using any one of the following six methods: (i) Row-Column-Depth method, (ii) Row-Depth-
 91 Column method, (iii) Column-Row-Depth method, (iv) Column-Depth-Row method, (v) Depth-Row-Column
 92 method and (vi) Depth-Column-Row method. Let us consider the three dimensional array of matrix size $4 \times 4 \times 4$
 93 as shown in Fig. ??.

94 7 Fig. 2: Sample three dimensional array

95 The above 3-D matrix is represented in a linear array of 2-D planes for easy understanding.

96 8 3D-RT using Row-Column-Depth (RCD) Approach

97 The 1D-RT of the four rows of the 0 th plane of the matrix X are $\{1,1,1,3\}$, $\{6,2,2,2\}$, $\{1,1,1,3\}$, $\{6,2,2,2\}$,
 98 $\{2,1,2,2\}$, $\{7,1,1,1\}$, $\{1,0,0,3\}$, $\{4,2,4,2\}$. The 1D-RT of the four rows of the 1 st plane are $\{1,0,1,0\}$, $\{2,2,0,0\}$,
 99 $\{1,2,3,0\}$, $\{6,2,4,0\}$, $\{0,0,2,3\}$, $\{5,1,5,1\}$, $\{3,0,3,3\}$, $\{9,3,3,3\}$. The 1D-RT of the four rows of the 2 nd plane are

100 $\{2,2,1,0\}, \{8,2,2,0\}, \{1,0,1,3\}, \{5,1,3,3\}, \{2,1,1,1\}, \{5,1,1,1\}, \{2,3,2,1\}, \{8,0,2,2\}$. The 1D-RT of the final plane of
101 four rows are $\{0,1,0,2\}, \{3,3,1,1\}, \{2,0,0,2\}, \{4,0,4,0\}, \{1,2,2,2\}, \{7,1,1,1\}, \{2,2,0,1\}, \{5,1,3,1\}$. Now the 3D array
102 of the Rajan Transform computed row wise is given by The 1D-RT of the four columns of the 0 th plane of the
103 matrix $X_r(k_1, k_2, k_3)$ are $\{6,6,7,4\}, \{23,3,3,1\}, \{2,2,1,2\}, \{7,1,1,1\}, \{2,2,1,4\}, \{9,3,3,1\}, \{2,2,1,2\}, \{7,1,1,1\}$.
104 The 1D-RT of the four columns of the 1 st plane are $\{2,6,5,9\}, \{22,8,6,0\}, \{2,2,1,3\}, \{8,2,2,0\}, \{0,4,5,3\}, \{12,2,6,4\},$
105 $\{0,0,1,3\}, \{4,2,4,2\}$. The 1D-RT of the four columns of the 2 nd plane are $\{8,5,5,8\}, \{26,0,6,0\}, \{2,1,1,0\}, \{4,2,2,0\},$
106 $\{2,3,1,2\}, \{8,2,2,0\}, \{0,3,1,2\}, \{6,4,2,0\}$. The 1D-RTs of the final plane of four columns are $\{3,4,7,5\}, \{19,1,5,3\},$
107 $\{3,0,1,1\}, \{5,3,3,1\}, \{1,4,1,3\}, \{9,5,1,1\}, \{1,0,1,1\}, \{3,1,1,1\}$. Now the 3D array of the Rajan Transform Row-
108 Column wise is given by The 1D-RT of the depth wise of four corresponding elements in four planes of
109 the matrix $X_{r,c}(k_1, k_2, k_3)$, (RT can compute starting with any element in 0 th plane, in this
110 example the RTs computed starting with row wise for easy understanding) are $\{2, 26, 19\}, \{90, 8, 6, 0\},$
111 $\{7, 8, 4, 5\}, \{24, 2, 6, 0\}, \{9, 12, 8, 9\}, \{38, 4, 4, 2\}, \{7, 4, 6, 3\}, \{20, 6, 2, 0\}, \{3, 8, 0, 1\}, \{22, 6, 10, 4\}, \{1, 2, 2, 3\}, \{8, 2, 2, 0\},$
112 $\{3, 2, 2, 5\}, \{12, 2, 4, 2\}, \{1, 2, 4, 1\}, \{8, 2, 4, 2\}, \{3, 6, 6, 5\}, \{20, 2, 4, 2\}, \{1, 2, 2, 3\}, \{8, 2, 2, 0\}, \{3, 6, 2, 1\}, \{12, 2, 6, 4\},$
113 $\{1, 4, 2, 1\}, \{8, 2, 4, 2\}, \{1, 0, 0, 3\}, \{4, 2, 4, 2\}, \{1, 0, 0, 1\}, \{2, 0, 2, 0\}, \{1, 4, 0, 1\}, \{6, 4, 4, 2\}$ and $\{1, 2, 0, 1\}, \{4, 2, 2, 0\}$. Now
114 the 3D array of the Rajan Transform Row-Column-Depth wise is given by 3D-RT using Row-Depth-Column
115 (RDC) Approach The 3D matrix using Row-Depth-Column Rajan Transform is Fig. 11 given in the next page,
116 shows that 3D-RT of the original 3-D matrix (Fig. 6) and the 3D-RT of the translated 3-D matrix (Fig. 10) are
117 the same. So, the invariance property of 3D-RT could be seen to be applicable to 3D images also. This amounts
118 to saying that 1D-RT, 2D-RT and 3D-RT could be reliably used in real world applications. Note that $X_{r,c,d}(k_1, k_2, k_3) = X_{r,d,c}(k_1, k_2, k_3)$, but
119

9 Cyclic Shift Invariance Property

120 Consider the sequence $x(n)=3,8,5,6,0,2,9,6$. Now, seven more cyclic shifted versions such as $x_{c1}(n)=6,3,8,5,6,0,2,9$; $x_{c2}(n)=9,6,3,8,5,6,0,2$; $x_{c3}(n)=2,9,6,3,8,5,6,0$; $x_{c4}(n)=0,2,9,6,3,8,5,6$; $x_{c5}(n)=6,0,2,9,6,3,8,5$;
121 $x_{c6}(n)=5,6,0,2,9,6,3,8$ and $x_{c7}(n)=8,5,6,0,2,9,6,3$ could be generated from $x(n)$. All the eight sequences
122 have the same $X(k)=39,5,13,9,13,1,7,5$, meaning Rajan Transform of a given sequence of length N would remain
123 invariant for N cyclically permuted sequences. This property is satisfied by higher order RT, that is, 2D-RT and
124 3D-RT also.

10 Graphical Inverse Invariance Property

125 Consider $x(n)=3,8,5,6,0,2,9,6$. Its graphical inverse is $x^{-1}(n)=6,9,2,0,6,5,8,3$. Now, one can generate seven more
126 cyclic shifted versions such as $x_{c1}^{-1}(n)=3,6,9,2,0,6,5,8$;
127 $x_{c2}^{-1}(n)=8,3,6,9,2,0,6,5$; $x_{c3}^{-1}(n)=5,8,3,6,9,2,0,6$; $x_{c4}^{-1}(n)=6,5,8,3,6,9,2,0$; $x_{c5}^{-1}(n)=0,6,5,8,3,6,9,2$;
128 $x_{c6}^{-1}(n)=2,0,6,5,8,3,6,9$ and $x_{c7}^{-1}(n)=9,2,0,6,5,8,3,6$. One can easily verify that all these eight sequences
129 have the same $X(k)=39,5,13, 9,13,1,7,5$, meaning Rajan Transform of a given sequence of length N would remain
130 invariant for N graphically inverted sequence and its cyclically permuted sequences. This property is satisfied by
131 higher order RT, that is, 2D-RT and 3D-RT also.^{1 2}

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²Multi-Dimensional Rajan Transform

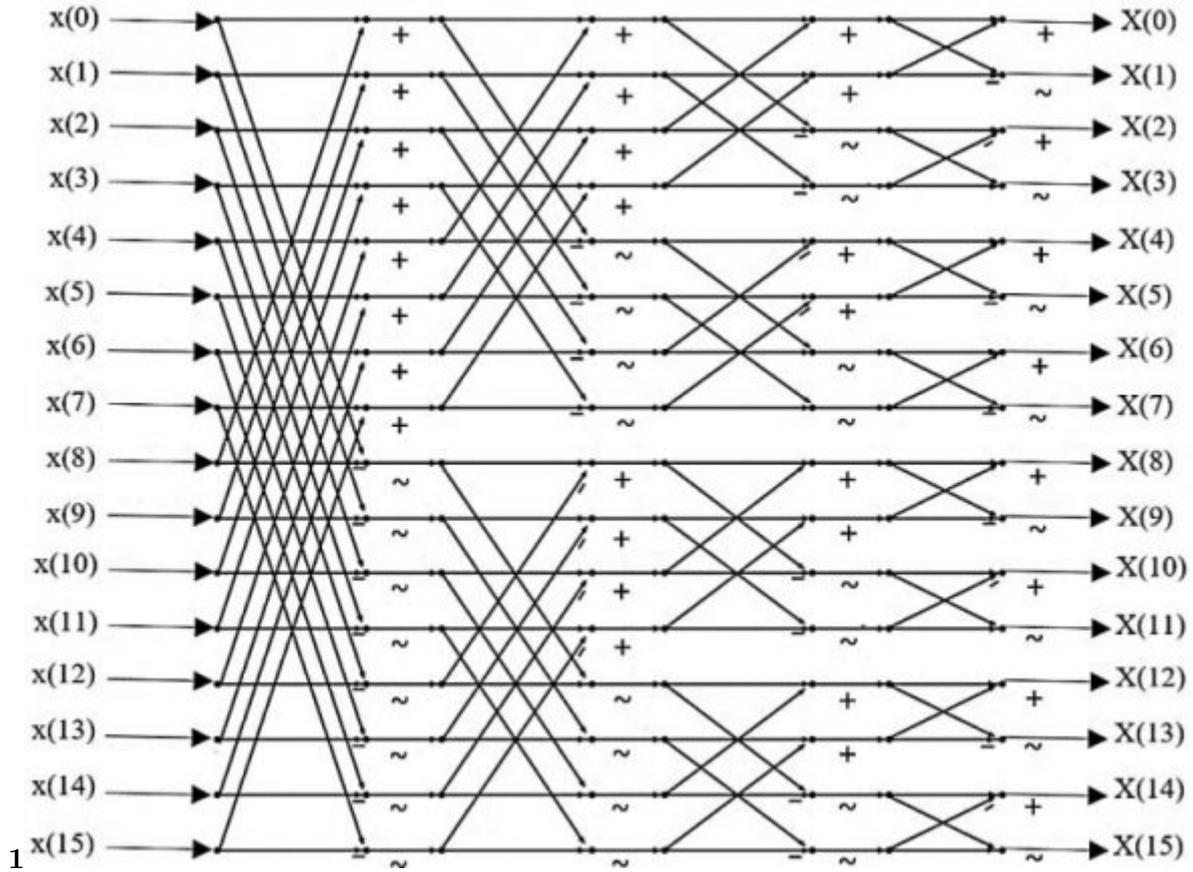


Figure 1: Fig. 1 :

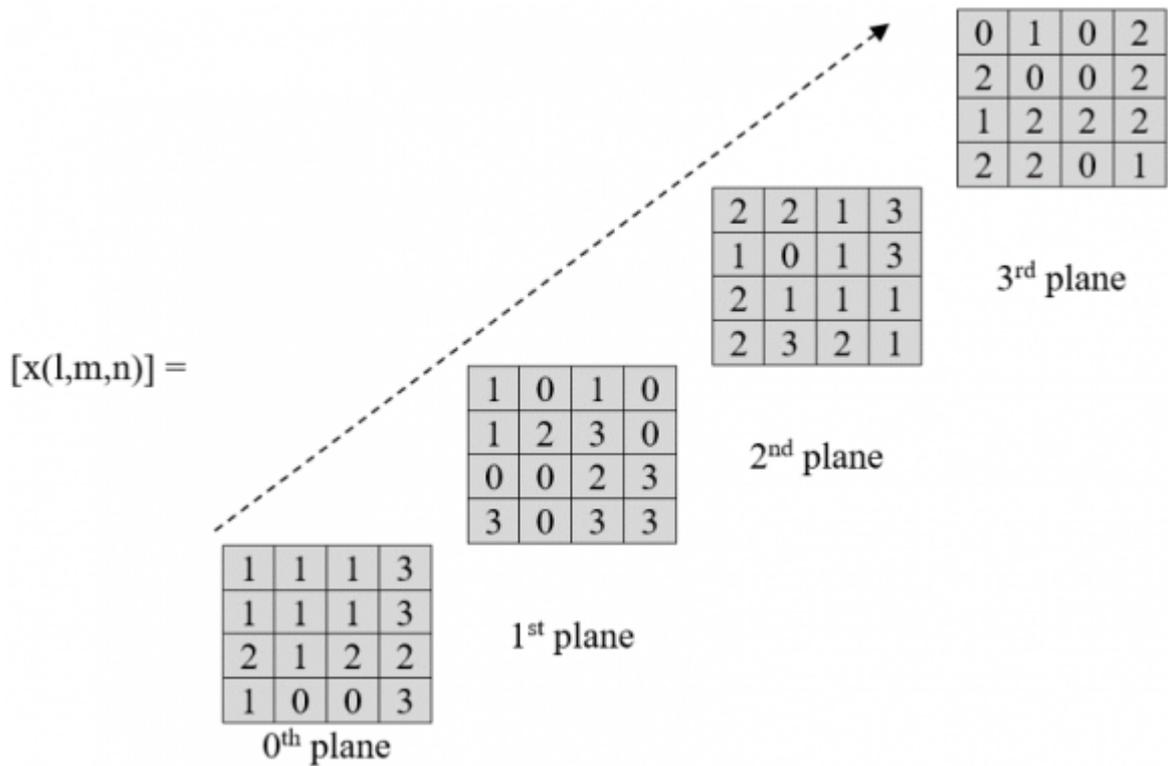


Figure 2:

$$5 \quad x(l, m, n) = \begin{pmatrix} \begin{matrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \\ 2 & 1 & 2 & 2 \\ 1 & 0 & 0 & 3 \end{matrix} & \begin{matrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \\ 3 & 0 & 3 & 3 \end{matrix} & \begin{matrix} 2 & 2 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 2 & 1 & 1 & 1 \\ 2 & 3 & 2 & 1 \end{matrix} & \begin{matrix} 0 & 1 & 0 & 2 \\ 2 & 0 & 0 & 2 \\ 1 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 \end{matrix} \\ \text{0}^{\text{th}} \text{ plane} & \text{1}^{\text{st}} \text{ plane} & \text{2}^{\text{nd}} \text{ plane} & \text{3}^{\text{rd}} \text{ plane} \end{pmatrix}$$

Figure 3: Fig. 5 :

$$6 \quad X_r(k_1, k_2, k_3) = \begin{pmatrix} \begin{matrix} 6 & 2 & 2 & 2 \\ 6 & 2 & 2 & 2 \\ 7 & 1 & 1 & 1 \\ 4 & 2 & 4 & 2 \end{matrix} & \begin{matrix} 2 & 2 & 0 & 0 \\ 6 & 2 & 4 & 0 \\ 5 & 1 & 5 & 1 \\ 9 & 3 & 3 & 3 \end{matrix} & \begin{matrix} 8 & 2 & 2 & 0 \\ 5 & 1 & 3 & 3 \\ 5 & 1 & 1 & 1 \\ 8 & 0 & 2 & 2 \end{matrix} & \begin{matrix} 3 & 3 & 1 & 1 \\ 4 & 0 & 4 & 0 \\ 7 & 1 & 1 & 1 \\ 5 & 1 & 3 & 1 \end{matrix} \\ \text{0}^{\text{th}} \text{ plane} & \text{1}^{\text{st}} \text{ plane} & \text{2}^{\text{nd}} \text{ plane} & \text{3}^{\text{rd}} \text{ plane} \end{pmatrix}$$

Figure 4: Fig. 6 :

$$7 \quad X_{r,c}(k_1, k_2, k_3) = \begin{pmatrix} \begin{matrix} 23 & 7 & 9 & 7 \\ 3 & 1 & 3 & 1 \\ 3 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{matrix} & \begin{matrix} 22 & 8 & 12 & 4 \\ 8 & 2 & 2 & 2 \\ 6 & 2 & 6 & 4 \\ 0 & 0 & 4 & 2 \end{matrix} & \begin{matrix} 26 & 4 & 8 & 6 \\ 0 & 2 & 2 & 4 \\ 6 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} 19 & 5 & 9 & 3 \\ 1 & 3 & 5 & 1 \\ 5 & 3 & 1 & 1 \\ 3 & 1 & 1 & 1 \end{matrix} \\ \text{0}^{\text{th}} \text{ plane} & \text{1}^{\text{st}} \text{ plane} & \text{2}^{\text{nd}} \text{ plane} & \text{3}^{\text{rd}} \text{ plane} \end{pmatrix}$$

Figure 5: Fig. 7 :

$$8 \quad X_{r,c,d}(k_1, k_2, k_3) = \begin{pmatrix} \begin{matrix} 90 & 24 & 38 & 20 \\ 12 & 8 & 12 & 8 \\ 20 & 8 & 12 & 8 \\ 4 & 2 & 6 & 4 \end{matrix} & \begin{matrix} 8 & 2 & 4 & 6 \\ 6 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 0 & 4 & 2 \end{matrix} & \begin{matrix} 6 & 6 & 4 & 2 \\ 10 & 2 & 4 & 4 \\ 4 & 2 & 6 & 4 \\ 4 & 2 & 4 & 2 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{matrix} \\ \text{0}^{\text{th}} \text{ plane} & \text{1}^{\text{st}} \text{ plane} & \text{2}^{\text{nd}} \text{ plane} & \text{3}^{\text{rd}} \text{ plane} \end{pmatrix}$$

Figure 6: Fig. 8 :

$$9 \quad X_{r,d,c}(k_1, k_2, k_3) = \begin{pmatrix} \begin{matrix} 90 & 24 & 38 & 20 & 18 & 6 & 12 & 12 \\ 4 & 2 & 12 & 6 & 2 & 2 & 0 & 4 \\ 10 & 6 & 4 & 4 & 10 & 2 & 4 & 4 \\ 0 & 4 & 2 & 2 & 2 & 2 & 0 & 4 \end{matrix} & \begin{matrix} 24 & 12 & 18 & 12 \\ 4 & 4 & 2 & 0 \\ 8 & 4 & 2 & 4 \\ 4 & 4 & 2 & 0 \end{matrix} & \begin{matrix} 8 & 0 & 10 & 4 \\ 4 & 0 & 2 & 0 \\ 4 & 0 & 2 & 4 \\ 0 & 0 & 2 & 0 \end{matrix} \\ \text{0}^{\text{th}} \text{ plane} & \text{1}^{\text{st}} \text{ plane} & \text{2}^{\text{nd}} \text{ plane} & \text{3}^{\text{rd}} \text{ plane} \end{pmatrix}$$

Figure 7: Fig. 9 :

136 .1 Acknowledgement

137 The first author expresses his gratitude to the second and third authors for introducing the fundamental notion
 138 of Rajan Transform for carrying out his Post-Doctoral research work.

139 .2 Dyadic Shift Invariance Property

140 Consider $x(n)=3,8,5,6,0,2,9,6$ and transpose its first half with the second half. The resulting sequence Td (2)
 141 $[x(n)]=0,2,9,6,3,8,5,6$ is the 2-block dyadic shifted version of $x(n)$. The symbol Td (2) denotes the 2-block dyadic
 142 shift operator. In the same manner, one would obtain Td (4) $[Td (2) [x(n)]]=9,6,0,2,5,6,3,8$ and Td (8) $[Td (4)$
 143 $[Td (2) [x(n)]]]=6,9,2,0,6,5,8,3$. Note that the graphical inverse of $x(n)$ is $x^{-1}(n) = (6,9,2,0,6,5,8,3)$ and it is
 144 the same as Td (8) $[Td (4) [Td (2) [x(n)]]]=6,9,2,0,6,5,8,3$. One can easily verify that all these dyadic shifted
 145 sequences have the same $X(k)$, that is, the sequence 39,5,13,9,13,7,5.

146 There is yet another way of dyadic shifting the input sequence $x(n)$ to Td (2) $[Td (4) [Td (8) [x(n)]]]$.
 147 Consider the sequence $x(n)=3,8,5,6,0,2,9,6$ and one can obtain Td (8) $[x(n)]=8,3,6,5,2,0,6,9$; Td (4) $[Td (8)$
 148 $[x(n)]]]=6,5,8,3,6,9, 2,0$ and Td (2) $[Td (4) [Td (8) [x(n)]]]=6,9,2,0,6,5,8,3$ as dyadic shifts. Note that Td (2) $[Td$
 149 $(4) [Td (8) [x(n)]]] = Td (8) [Td (4) [Td (2) [x(n)]]]$. This property is satisfied by higher order RT, that is, 2D-RT
 150 and 3D-RT also.

151 .3 Dual Class Invariance Property

152 Given a sequence $x(n)$, one can construct another sequence $y(n)$ consisting of at least one number which is not
 153 present in $x(n)$ such that $X(k)=Y(k)$. In such a case, $y(n)$ is called the 'dual' of $x(n)$. Consider two sequences
 154 $x(n)=2,4,2,2$ and $y(n)=3,1,3,3$. Then $X(k)=Y(k)=10,2,2,2$. An underlying theorem to characterize a sequence
 155 of length $N=2^n$ to pair up with a dual sequence is "A sequence is said to have a dual if and only if its CPI
 156 is an even number and is divisible by $N/2$ ". This theorem advocates a necessary condition but not a sufficient
 157 condition. For example, consider the sequence $x(n)=6,8,2,0$. This indeed satisfies the theorem. That is, its CPI
 158 is 16 and the value of $CPI/(N/2)$ is 8. Now its dual is computed as $y(n)=2,0,6,8$, which is not a dual of $x(n)$ as
 159 per the definition. In such cases, they are called 'self dual' pairs. Some of the properties of dual sequences are:
 160 (i) if $y(n)$ is a dual of $x(n)$, then $x(n)$ is also called the dual of $y(n)$. Hence the pair $\langle x(n),y(n) \rangle$ is called 'dual
 161 pair'; (ii) dual of a sequence, say $y(n)$ will necessarily exhibit geometric symmetry together with the original
 162 sequence $x(n)$; (iii) each dual pair has a value called 'Differential Mean' (DM), which is equal to $(|x(i)-y(i)|)/2$
 163 ; $0 \leq i \leq (N-1)$ about which the dual sequences are 'flip' symmetric. DM could be a real number. This property is
 164 satisfied by higher order RT, that is, 2D-RT and 3D-RT also.

165 .4 Scalar Property

166 Let $x(n)$ be a number sequence and α be a scalar. Then the RT of $\alpha x(n)$ will be $\alpha X(k)$, where $X(k)$ is the RT of
 167 $x(n)$. For example, let us consider a sequence $x(n)=1,3,1,2$ and a scalar α of value 2. Now the RT $X(k)$ of $x(n)$
 168 is 7,3,1,1. RT of $\alpha x(n)=2,6,2,4$ is 14,6,2,2 which is nothing but $\alpha X(k)$. This property is satisfied by higher order
 169 RT, that is, 2D-RT and 3D-RT also.

170 .5 Linearity Property

171 In general, RT does not satisfy the linearity property for all sequences. It was observed that for a pair $x(n)$ and
 172 $y(n)$ which are number sequences either in the increasing order or in the decreasing order, the linearity property
 173 works. That is, for $\alpha x(n)+\beta y(n)$ where α and β are scalars and $x(n)$ and $y(n)$ are two number sequences either
 174 in the increasing or decreasing order, the RT will be $\alpha X(k)+\beta Y(k)$, where $X(k)$ and $Y(k)$ are respectively the
 175 RTs of $x(n)$ and $y(n)$. This property is satisfied by higher order RT, that is, 2D-RT and 3D-RT also.

176 [Thiagarajan (2020)] *PRC/Admn/PDF-Thiagarajan/2019-1, Pattern Recognition Division, Thiagarajan . Au-*
 177 *gust 10, 2020. Pentagram Research Centre, Hyderabad, India. (Multi-Dimensional Rajan Transforms and*
 178 *Their Applications)*