A Linear Algorithm for Convex Drawing of a Planar Graph

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Abstract - A straight line drawing of a planar graph is called a convex drawing if the boundaries of all faces of that graph are drawn as convex polygon. A graph is planar if it has at least one embedding in the plane such that no two edges intersect at any point except at their common end vertex. Not all planar graphs have convex drawing. In this thesis, we study the characteristics of convex drawing of a planar graph. We develop a method for examining whether a face is drawn as a convex polygon or not.

Finally, using that method we develop a linear algorithm for examining whether a planar graph has a convex drawing or not.

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1. Introduction

Some planar graphs can be drawn in such a way that each edge is drawn as a straight line segment and each face is drawn as a convex polygon, as illustrated in Figure 3.1. Such a drawing is called a convex drawing. The drawings in Figs. 3.2 are not convex drawings.

Figure 1.1: Convex drawing of planar graph

Although not every planar graph has a convex drawing, Tutte showed that every 3-connected planar graph has a convex drawing, and obtained a necessary and sufficient condition for a plane graph to have a convex drawing [5]. Furthermore, he gave a "barycentric mapping" method for finding a convex drawing of a plane graph, which requires solving a system of $O(n)$ linear equations [6]. The system of equations can be solved either in $O(n^3)$ time and $O(n^2)$ space using the ordinary Gaussian elimination method, or in $O(n^{1.5})$ time and $O(n \log n)$ space using the sparse Gaussian elimination method [LRT79]. Thus the barycentric mapping method leads to an $O(n^{1.5})$ time convex drawing algorithm for planar graphs. In this chapter we first give a lemma for a face is drawn as convex polygon or not. Then using that lemma finally we device a linear time algorithm to examine whether a planar graph has convex drawing or not.

Figure 1.2: Non convex drawing of planar graph
II. Definition

By extensively examining the characteristics of convex drawing of a planar graph we derive a lemma for examining whether a planar graph has convex drawing or not. Before introducing the lemma we need to define some terms.

a) Convex Drawing of Planar Graph

A straight line drawing of a planar graph $G$ is called a convex drawing if the boundaries of all faces of $G$ are drawn as convex polygons [8]. Figure 3.3 depicts a convex drawing of a planar graph.

![Figure 2.1: Convex drawing of a planar graph](image)

b) Face

If $G$ is a planar graph, then any plane drawing of $G$ divides the plane into regions, called faces [9]. That is, a face is an area bounded by the edges. One of these faces is unbounded, and is called the infinite face.

If $f$ is any face, then the degree of $f$ (denoted by $\deg f$) is the number of edges encountered in a walk around the boundary of the face $f$. If all faces have the same degree ($g$, say), the $G$ is face-regular of degree $g$. For example, the following graph $G$ depicts in Figure 3.4 has six faces, $f_6$ being the infinite face.

![Figure 2.2: A planar graph with six faces](image)

It is easy to see from above graph that $\deg f_1=3$, $\deg f_2=4$, $\deg f_3=3$, $\deg f_4=7$, $f_5=4$. Note that the sum of all the degrees of the faces is equal to twice the number of edges in the graph, since each edge either borders two different faces (such as $bg$, $cd$, and $cf$) or occurs twice when walk around a single face (such as $ab$ and $gh$). The Euler's formula relates the number of vertices, edges and faces of a planar graph. If $n$, $m$, and $f$ denote the number of vertices, edges, and faces respectively of a connected planar graph, then we get $n-m+f = 2$. The Euler formula tells us that all plane drawings of a connected planar graph have the same number of faces namely, $2+m-n$.

III. Theorem

(Euler's Formula) Let $G$ be a connected planar graph, and let $n$, $m$ and $f$ denote, respectively, the numbers of vertices, edges, and faces in a plane drawing of $G$. Then $n-m+f = 2$.

Proof We employ mathematical induction on edges, $m$. The induction is obvious for $m=0$ since in this case $n=1$ and $f=1$. Assume that the result is true for all connected plane graphs with fewer than $m$ edges, where $m$ is greater than or equal to 1, and suppose that $G$ has $m$ edges. If $G$ is a tree, then $n=m+1$ and $f=1$ so the desired formula follows. On the other hand, if $G$ is not a tree, let $e$ be a cycle edge of $G$ and consider $G-e$. 

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The connected plane graph $G-e$ has $n$ vertices, $m-1$ edges, and $f-1$ faces so that by the inductive hypothesis,

$$n - (m-1) + (f-1) = 2$$

Which implies that?

$$n - m + f = 2$$

(Proved)

(a) **Convex polygon**

A convex polygon is a simple polygons whose interior is a convex set [8]. The following properties of a simple polygon are all equivalent to convexity:

- Every internal angle is less than 180 degrees.
- Every line segment between two vertices remains inside or on the boundary of the polygon.

A simple polygon is strictly convex if every internal angle is strictly less than 180 degrees. Figure 3.5(a) and (b) depicts a convex and non-convex polygon respectively.

(b) **Lemma**

A face is drawn as convex polygon if and only if the cross products of adjacent edges of each vertex of that face are same sign.

**Proof**

Let, a face is assumed to be described by $N$ vertices ordered by,

$$V_0(x_0, y_0), V_1(x_1, y_1), V_2(x_2, y_2), \ldots, V_n(x_n, y_n)$$

**cross product** = $((x_i - x_{i+1}), (y_i - y_{i+1})) \times ((x_{i+1} - x_i), (y_{i+1} - y_i))$

$$= (x_i - x_{i+1}) * (y_{i+1} - y_i) - (y_i - y_{i+1}) * (x_{i+1} - x_i)$$

Figure 3.6 (a) and (b) depicts a face in clockwise and anti-clockwise vertex ordering respectively. A simple test of vertex ordering for examining a face is drawn as convex polygon is based on considerations of the cross product between adjacent edges of each vertex of that face. If the cross product is positive then it rises above the plane ($z$ axis up out of the plane) and if negative then the cross product is into the plane.
Figure 3.2 (a) and (b) depicts the cross product sign of adjacent edges of each vertex face depicts in figure 3.2 (a) and (b) respectively.

Figure 3.3 : (a) Cross products sign of adjacent edges in clockwise direction of convex face and (b) Cross products sign of adjacent edges in anti-clockwise direction of convex in the case of non convex face the cross product sign of adjacent edges of each vertex of that face depicts in figure 3.4(a) and (b)

Figure 3.4 : (a) Cross products sign of adjacent edges in clockwise direction of non-convex face and (b) Cross products sign of adjacent edges in anti-clockwise direction of non-convex face

A non-convex face has mixture of cross products sign of adjacent edges of each vertex of that face. Hence, a face is drawn as convex polygon if and only if the cross products of adjacent edges of each vertex of that face are same sign.

[Proved]
Start

Input a planar graph

Any curved edges?

yes

Replace curved edges by straight line edges.

no

Compute no. faces f of that graph and vertex set V_n that formed each face F_i.

Check each face F_i is convex polygon or not.

Compute cross products C_j of adjacent edges of vertex v_j of face F_i.

Is C_j = 0?

yes

Has no convex drawing of that planar graph.

no

Are C_j and C_{j-1} same sign?

yes

Has convex drawing of that planar graph.

no

j <= n?

yes

Exit.

no

i <= f?

yes

no

no
(d) Algorithm: Convex_Drawing (G)

Begin

Step 1: Check input planar graph has curved edges.
   if (curved edges)
      then replace curved edges by straightline edges.
   else
      go to step 2.
   end if

Step 2: Compute no. faces f of that graph and vertex set V_n that formed each face F_i.
   i.e., F_i = \{v_0, v_1, v_3, \ldots \ldots , v_m\} where i=1 to f and m=0 to n.

Step 3. 3(a): Check each face F_i is convex polygon or not.
   for each face F_i where i=1 to no. of face f do
      for j=0 to no. of vertex n to form that face F_i do
         compute cross product C_j of adjacent edges of vertex v_j.
         cross product of adjacent edges of vertex v_j
         C_j = (X_j-X_{j-1})*(Y_{j+1}-Y_j)-(Y_j-Y_{j-1})*(X_{j+1}-X_j).
         if (cross product C_j = 0)
            then go to step 5.
         else
            go to step 3(b).
         end if
      end for.
   end if.

3(b): if (j=0)
   then increment j by 1.
   else
      check cross product C_j and C_{j-1} are same sign.
      if (cross product C_j and C_{j-1} are same sign)
         then increment j by 1.
      else
         go to step 5.
      end if
   end if
end for.

Step 4: Has Convex drawing of that planar graph.
   go to step 6.

Step 5: Has no convex drawing of that planar graph.
   go to step 6.

Step 6: Exit.
End

IV. Conclusions

In this thesis we have studied the convex drawing of a planar graph. Not every planar graph has convex drawing. The results of this thesis are summarized as follows:

- We have derived a method for determining whether a face is drawn as convex polygon or not.
- Finally, using that method we develop a linear time algorithm for examining whether a planar graph has a convex drawing or not.

Some interesting directions in which the future research works can be done as follows:

- We develop a linear time algorithm for examining whether a planar graph has a convex drawing or not. One can develop an algorithm for converting non-convex drawing of a planar graph to convex drawing.
- One can develop a convex grid drawing of a planar graph on an (n-2) x (n-2) grid.

References Références Referencias

1. Md. Shaidur Rahman, Takao Nishizeki “Planar Graph Drawing”.

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