Generalized Hadamard Matrices from Generalized Orthogonal Matrix

Sudha Singh[1],M. K. Singh[2], D. K. Singh[3]

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***Abstract*-A new generalization of matrix orthogonality is introduced. It is shown thatfrom generalized orthogonal matrices some known as well as a few new complex H- matrices with circulant blocks can be obtained. The orders of new complex H-matrix are 26, 36, 50 and 82.**

IndexTerms- Circulant matrix, Hadamard matrix, Generalization of Hadamard matrix, Quaternion, Associative algebra of matrices, generalized orthogonal matrix.

I. Introduction

First we recall the following definitions:

Circulant Matrix: It is an $n \times n$ matrix of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 \ldots\ldots a_n \\ a_n & a_1 & a_2 \ldots\ldots a_{n-1} \\ a_{n-1} & a_n & a_1 \ldots\ldots a_{n-2} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_2 & a_3 & a_4 \ldots\ldots a_1 \end{pmatrix}$$

which is denoted as $Circ(a_1 a_2 a_3 \ldots\ldots a_n)$.

Hadamard matrix (or an H-matrix): It is an $n \times n$ matrix H with entries +1, -1 such that $HH^T = nI_n$, where I_n is the $n \times n$ identity matrix.

Complex H-matrix: It is an $n \times n$ matrix $H = [H_{ij}]$, where H_{ij} are complex numbers with $|H_{ij}| = 1$ for i, j = 1, 2, . . . , n, satisfying $HH^* = nI$, where I is the identity matrix and H^* denotes the Hermitian transpose[9] of H. A complex H-matrix is called dephased if elements of its first row and column are 1.

Butson H-matrix: It is an $n \times n$ complex Hadamard matrix with elements belonging to theset of m^{th} roots of 1 and is denoted as $BH(m,n)$.

Unimodular complex H-matrix: It is an $n \times n$ complex H-matrix whose elements are of the form $EXP(i\theta)$. An _

m-parameteraffine complex Hadamard family(or orbit) H(R) stemming from a dephased $n \times n$ complex Hadamard matrix H is the set of matrices A satisfying $AA^* = nI$, associated with an m-dimensional subspace R of a space of all real $n \times n$ matrices with zeros in the first row and column,

Weighing matrix W(n,w): A W(n,w) of order n and weight w is an $n \times n$ $(0,1,-1)$-matrix such that $WW^T = wI$, where w is a positive integer.

Conference matrix: It is a weighing matrix $W(n, n-1)$ with0 occurring only on the diagonal.

Quaternion: A number of the form q = a1 + bi + cj + dk, where $i^2 = j^2 = k^2 = -1$, k = ij = -ji, a, b, c, d are real numbers, is called a quaternion or a hypercomplex number. q reduces to a complex number when c = d = 0 and to a real number when b = c = d = 0. If 1, i, j, k are taken as 2×2 matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ and $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ respectively we get two dimensional complex matrix representation of the quaternion q = $\begin{pmatrix} a - ci & b + di \\ -b + di & a + ci \end{pmatrix}$.

Replacing 1 by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and i by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the matrices 1, i, j, k we get four dimensional real matrix representation of the quaternion q = $\begin{pmatrix} a & -c & b & d \\ c & a & -d & b \\ -b & d & a & c \\ -d & -b & -c & a \end{pmatrix}$.

The Hermitian conjugate of a quaternion q = a1 + ib + jc + kd is $\bar{q}$ = a1 - ib - jc – kd and the modulus of q is $|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$.

Associative algebra of matrices: Let R be a ring of complex numbers and A be a vector space of vxv matrices over R with basis matrices I =$A_0, A_1, \ldots A_m$. A is called an associative algebra with unity I if they satisfy

$A_i A_j = \sum_{k=0}^{m} p_{ij}^k A_k$, where p^k_{ij} are in general complex numbers (1)

About[1]-PG Department of Computer Secience and Engg, Bengal College of Engg and Technology, Durgapur-713212(West Bengal),India;sudha_2k6@yahoo.com
About[2]- PG Department of Mathematics, Ranchi University, Ranchi-834008(Jharkhand), India;mithileshkumarsingh@gmail.com
About[3]- Department of Electronics and Communications BIT Sindri, Dhanbad-(828123) (Jharkhand), India, dksingh_bit@yahoo.com.

In what follows we assume that the elements of A_i are only 0, 1 and –1 and p^k_{ij} are integers called multiplication coefficients of the algebra.
Example 1 Algebra of quaternions spanned by the matrices 1, i, j, k of order 2 or 4 given in 1.4.
Example 2 Algebra of circulant matrices spanned by $A_i = w^i = [circ(0,1, 0,. . .0)]^i$,
i= 1,2,…m satisfying $A_iA_j= A_{i+j}$, where i+j is the addition mod m. We also consider
the algebra spanned by the direct product of circulant matrices of different orders viz.
$w_s^i \times w_t^j$, where w_s= circ(01,0,. . .0) of order s.
Example 3 Bose-Mesner algebra spanned by (0, 1) symmetric commuting matrices A_i satisfying
$A_0+A_1 + ...+A_m = J_v$ (all 1 matrix) and (1), where $A_0= I_v$,and p^k_{ij} are nonnegative integers.
(A_0, A_1 ...A_m) defines an m-class association scheme(or m-AS) with parameters p^k_{ij} .
A 2-AS is also called strongly regular graph and its parameters satisfy
$p^0_{ii} = n_i$, i=1, 2, and p^k_{ij} . satisfy
$p^k_{ij} = p^k_{ji}$, $p^j_{i0}= \delta_{ij}$, n_1+n_2= v-1, and $p^i_{j1} + p^i_{j2}= n_j- \delta_{ij}$., i,j=1,2, where δij=0 for i≠j and
δij=1 for i=j. (see Raghavarao[4]).

Generalized orthogonal matrix (GOM): Let A be an $m \times n$ matrix whose entries are the element of an associative algebra of matrices over a ring of complex numbers. The conjugate of an element a = $\sum_{g\in G} \alpha_i A_i \in A$ will be denoted by $\bar{a} = \sum_{g\in G} \overline{\alpha_i} A_i^T$, where $\overline{\alpha_i}$ is the complex conjugate of α_i and T stands for transpose.

A will be called a generalized orthogonal matrix if the dot product of any two rows

$$R_i.R_j = (a_{i1}, a_{i2} ..., a_{in})(b_{j1}, b_{j2} ..., b_{jn})$$

$$= \sum_{k=1}^{n} a_{ik} \overline{b_{jk}} =$$

$$\begin{cases} \lambda J, \text{ if } i \neq j \\ \lambda_0 I + \lambda_1 \sum_{i=1}^{m} A_i \quad \text{if } \quad i = j, \end{cases}$$

where λ, λ_0, λ_1 are integers independent of i and j. Here λ, λ_0, λ_1 will be called parameters of orthogonal matrix A.

The purpose of this paper is to show that notion of generalized orthogonal matrix provides a general frame-work for constructing several classical real H-matrices of Paley[3] Williamson[7] and Ito [1] as well as some new Butson H-matrices and GDG H-matrices through special methods or computer search.We also identify some Butson H-matrices which admit non-Dita-type affine complex Hadamard family(or orbit)(vide sz¨oll″osi [5]) Such matrices are recently being used in quantum information theory and quantum tomography.Notations: The circulant matrix circ(0,1,0,…,0)will be denoted as w_n . The direct product of w_m, w_n will be denoted as $w_m x w_n$.

II. Construction of complex H-matrices from generalized orthogonal matrices

A. *Construction of some H-matrices with circulant blocks*

Construction of certain Paley type-I H-matrices [for definition see page 12,chapter 2 of 10]

Theorem I : Let $p = 4t - 1$ be a prime. If $(d_1, d_2, d_3, ..., d_k) \bmod p$ be a difference set[11,10], then GO-matrix $A = [w_p^{d_1} + w_p^{d_2} + ... + w_p^{d_k}]$, where $w_p = Circ(0,1,0,0,...,0)_p$ gives the core of a H-matrix of order 4t, if we replace 0 by -1 in A.

Construction of H-matrices of Williamson's form[10]

Williamson H-matrix of order $4(2m+1)$ is itself a 1×1 generalized orthogonal matrix
H = 1 x A + i x B + j x C + k x D, where 1, i, j, k are 4x4 matrix representation of basic quaternions,

$$1 = I_4, i = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, j = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \text{ and } A,$$

B, C, D are (+1,-1) suitable linear combinations of (0,1)-circulant matrices $W_1, W_2, W_3, ..., W_m$ of order n, that span a Bose-mesner algebra(see Raghavarao[4], for details of Bose-mesner algebra).

Construction of whiteman's type H–matrices of order (pq+1) where p and q are twin primes (vide whiteman[8])

Theorem II :Let p and q be two primes, q = p+2.
An 1x1 generalized orthogonal matrix

$$A = [(1 + w_p + w_p^2 + ... + w_p^{p-1}) \times I_q + (w_p \times w_q)^{d^1} + (w_p \times w_q)^{d^2} + ... + (w_p \times w_q)^{d^k}]$$

where

$$w_p = Circ(0,1,0,0,...,0)_p$$

$$w_q = Circ(0,1,0,0,...,0)_q$$

$$\text{and d satisfies } d^k \equiv 1 (\bmod p),$$

$$d^k \equiv 1 (\bmod q), \text{ k} = \frac{(p-1)(q-1)}{2}$$

gives the core of H-matrix of order (pq+1), if we replace 0 by -1 everywhere in A.

Construction of an H-matrix of order 36

We consider on 3x7 rectangular generalized orthogonal matrix A=

$$\begin{pmatrix} \omega+\omega^4 & \omega^2+\omega^3 & 0 & 0 & I_5 & I_5 & I_5 \\ I_5 & I_5 & \omega+\omega^4 & \omega^2+\omega^3 & I_5 & 0 & 0 \\ 0 & 0 & I_5 & I_5 & I_5 & \omega+\omega^4 & \omega^2+\omega^3 \end{pmatrix}, \text{where}$$

$\omega = circ\ (01000\ ), 0 = 5\times 5$ null matrix and I_5 = unit matrix.

Then replacing 3 by 0 and 0 by -1 in A^TA, we get the core of a H-matrix (see Horadam[10] for definition and details of core) of order 36.

B. H-matrices from generalized orthogonal matrices arising from BIBDs (see Hall [12] for BIBDs)

Theorem III: Existence of a BIBD with parameters $v=2n^2-n, b=4n^2-1, r=2n+1, k=n, \lambda=1$ implies the existence of an H- matrix of order $4n^2$.

Method of construction: Let N be the incidence matrix of BIBD with parameters mentioned in theorem III. N^tN is a $b\times b$ square matrix. Let A be the (1,-1) matrix obtained from N^tN by replacing diagonal entries by -1, 1 by 1 and 0 by -1. Then A is a 1×1 generalized orthogonal matrix and $\begin{pmatrix} -1 & e \\ e^t & A \end{pmatrix}$ is a H-matrix of order $4n^2$ where e is $1\times(4n^2-1)$ matrix of 1's, e^t is the transpose of e.

Example 4: We consider the BIBD
Parameters: v=6, b=15, r=5, k=2, λ=1.

Let N be the incidence matrix of BIBD with given parameters.

Its dual N' is

$$\begin{pmatrix} 000011 \\ 110000 \\ 101000 \\ 100100 \\ 100010 \\ 100001 \\ 011000 \\ 010100 \\ 010010 \\ 010001 \\ 001100 \\ 001010 \\ 001001 \\ 000110 \\ 000101 \end{pmatrix}$$

The product of N and N' is

$$\begin{pmatrix} 2111111110\ 00000 \\ 1211110001\ 11000 \\ 1121101001\ 00110 \\ 1112100100\ 10101 \\ 1111200010\ 01011 \\ 1100021111\ 11000 \\ 1010012111\ 00110 \\ 1001011210\ 10101 \\ 1000111120\ 01011 \\ 0110011002\ 11110 \\ 0101010101\ 21101 \\ 0100110011\ 12011 \\ 0011001101\ 10211 \\ 0010101011\ 01121 \\ 0001100110\ 11112 \end{pmatrix}$$

$= 2A_0 + 1A_1 + 0A_2$

where $I = A_0, A_1, A_2$ span Bose-Mesner algebra. From NN' we can obtain a 1×1 generalized orthogonal matrix A by replacing 2 by 0 and 0 by -1. Adjoining a row of all 1's and a column of all 1's we get the following 16×16 H-matrix

$$\begin{pmatrix}
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1
\end{pmatrix}$$

This matrix attains an affine orbit by lemma 3.4 (see SZOLLOSI [5])

Example 5: We consider the BIBD
Parameters: v =15, b=35, r=7, k=3, λ=1.

Let N be the incidence matrix of BIBD with given parameters. Its dual N' is

Fig 1: MATRIX-1

The product of N and N' is

Fig 2: MATRIX-2

$=3A_0+1A_1+0A_2$

where $I=A_0,A_1,A_2$ span Bose-Mesner algebra. From NN' we can obtain a 1×1 generalized orthogonal matrix A by replacing 3 by 0 and 0 by -1. Adjoining a row of all 1's and a column of all 1's we get the following 36×36 H-matrix

Fig 3: MATRIX-3

This matrix attains an affine orbit by lemma 3.4 (see SZOLLOSI [5]).

C. Construction of some new Butson H-matrices

Example 6: Butson H-matrices can be constructed from the following circulant representation of some generalized orthogonal matrices:

(i) $A_5=[w+w^4 \quad w^2+w^3]$,

where $w=w_5=circ(01000)$

(ii) $A_{13}=[w+w^3+w^9 \quad w^2+w^5+w^6]$, where $w=w_{13}=circ(010\ldots0)$ of order13

(iii) $A_5=[circ(1+w^4,w,0,0,1)\ \mathrm{circ}(1+w^3,0,w^2,1,0)]$, $where \mathrm{w}=\mathrm{w}_5=circ(01000)$

(iv) $A_{41}=[w+w^{37}+w^{16}+w^{18}+w^{10} \quad w^8+w^9+w^5+w^{21}+w^{39}]$.

where $w=circ(01\ldots0)$ of order 41.

Method of Construction: Let A be any of the matrices above in (i), (ii),(iii) or (iv).

a) Obtain the symmetric square matrix A^tA.

b) **In A^tA replacing diagonal element by 1, 0 by –i and 1 by i, we get Butson** H-matrices $BH(4,2n)$ for 2n= 10, 26, 50 and 82.

c) In A^tA replacing diagonal elements by 0, we get a conference matrix.

Remark 1 The matrices of above orders constructed from circulant matrices of order 5, 13 and 41 appears to be different from those arising from well-known constructions from Galois fields of order 25, 49 and 81.

Remark 2: Since Hadamard matrices obtained in the above theorem are derivable from conference matrices, each matrix A is non Dita-type and admits an affine family of complex Hadamard matrices of at least one parameter which contains A (vide sz¨oll˝osi's theorems 4.1 and 4.2 in [5]).

Remark 3: In the recent catalogue [6] only Dit¸˘a-type matrices were considered in dimensions N = 10 and 14. Sz¨oll˝osi [5] presents non Dita-type matrix of order 10. In view of Theorem 4.1 and of 4.2 of sz¨oll˝osi we can now present new parametric families of non Dita-type complex Hadamard matrices of order 26, 50, 82.

(1) $BH(4,26)$

A Butson H-matrix of order 26 obtained by the above method is :

Fig 4: MATRIX-4

(2) $BH(4,50)$

A Butson H-matrix of order 50 obtained by the above method is :

Fig 5: MATRIX-5

D. Some Butson H-matrices $BH(m,n)$ for $m=3,6$.

Following Butson H-matrices are obtained from suitable generalized orthogonal matrices

(i) $BH(3,6):\begin{pmatrix}1&1&1&1&1&1\\1&&&&&\\1&&A&&&\\1&&&&&\\1&&&&&\\1&&&&&\end{pmatrix}$ where the core A is

given by $A=[I_5+w(w_5+w_5^4)+w^2(w_5^2+w_5^3)]$,

where $w_5=circ(01000)$ and w is an imaginary cube root of unity.

(ii) BH(3,9)

$$\begin{pmatrix}
1 & w & w & w & w & w^2 & w^2 & w^2 & w^2\\
w & 1 & w & w^2 & w^2 & w & w & w^2 & w^2\\
w & w & 1 & w^2 & w^2 & w^2 & w^2 & w & w\\
w & w^2 & w^2 & 1 & w & w & w^2 & w & w^2\\
w & w^2 & w^2 & w & 1 & w^2 & w & w^2 & w\\
w^2 & w & w^2 & w & w^2 & 1 & w & w & w^2\\
w^2 & w & w^2 & w^2 & w & w & 1 & w^2 & w\\
w^2 & w^2 & w & w & w^2 & w & w^2 & 1 & w\\
w^2 & w^2 & w & w^2 & w & w^2 & w & w & 1
\end{pmatrix}$$

(iii) BH(6, 7)

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -w^2 & -1 & -1 & w & -w & -w \\ 1 & -1 & -w^2 & -1 & -w & w & -w \\ 1 & -1 & -1 & -w^2 & -w & -w & w \\ 1 & w & -w & -w & -w^2 & -1 & -1 \\ 1 & -w & w & -w & -1 & -w^2 & -1 \\ 1 & -w & -w & w & -1 & -1 & -w^2 \end{pmatrix}$$

III. Conclusion

Butson H-matrices are constructed from generalized orthogonal matrices by replacement or minor changes. During constructions we get new complex H-matrices of orders 26, 36, 50 and 82, which is not equivalent to existing complex Hadamard matrices of same order. We hope that in future generalized orthogonal matrices will provide insights to construct more matrices of combinatorial and practical interests.

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Fig 1: Matrix-1

$$\begin{pmatrix} 0100110000 & 00000 \\ 1010001000 & 00000 \\ 0101000100 & 00000 \\ 0010100010 & 00000 \\ 1001000001 & 00000 \\ 0011010000 & 00000 \\ 0001101000 & 00000 \\ 1000100100 & 00000 \\ 1100000010 & 00000 \\ 0110000001 & 00000 \\ 0000001001 & 10000 \\ 0000010100 & 01000 \\ 0000001010 & 00100 \\ 0000000101 & 00010 \\ 0000010010 & 00001 \\ 0000000110 & 10000 \\ 0000000011 & 01000 \\ 0000010001 & 00100 \\ 0000011000 & 00010 \\ 0000001100 & 00001 \\ 1000010000 & 10000 \\ 0100001000 & 01000 \\ 0010000100 & 00100 \\ 0001000010 & 00010 \\ 0000100001 & 00001 \\ 1000000000 & 01001 \\ 0100000000 & 10100 \\ 0010000000 & 01010 \\ 0001000000 & 00101 \\ 0000100000 & 10010 \\ 1000000000 & 00110 \\ 0100000000 & 00011 \\ 0010000000 & 10001 \\ 0001000000 & 11000 \\ 0000100000 & 01100 \end{pmatrix}$$

Fig 2: Matrix-2

$$\begin{pmatrix}
3011011111 & 0100100110 & 1100101001 & 01001 \\
0301111111 & 1010000011 & 1110010100 & 10100 \\
1030111111 & 0101010001 & 0111001010 & 01010 \\
1103011111 & 0010111000 & 0011100101 & 00101 \\
0110311111 & 1001001100 & 1001110010 & 10010 \\
1111131001 & 0100100110 & 1011000110 & 00110 \\
1111113100 & 1010000011 & 0101100011 & 00011 \\
1111101310 & 0101010001 & 1010110001 & 10001 \\
1111100131 & 0010111000 & 1101011000 & 11000 \\
1111110013 & 1001001100 & 0110101100 & 01100 \\
0100101001 & 3011011111 & 1100101001 & 00110 \\
1010010100 & 0301111111 & 1110010100 & 00011 \\
0101001010 & 1030111111 & 0111001010 & 10001 \\
0010100101 & 1103011111 & 0011100101 & 11000 \\
1001010010 & 0110311111 & 1001110010 & 01100 \\
0011000110 & 1111131001 & 1011001001 & 00110 \\
0001100011 & 1111113100 & 0101110100 & 00011 \\
1000110001 & 1111101310 & 1010101010 & 10001 \\
1100011000 & 1111100131 & 1101000101 & 11000 \\
0110001100 & 1111110013 & 0110110010 & 01100 \\
1100110110 & 1100110110 & 3000011001 & 10110 \\
1110001011 & 1110001011 & 0300011100 & 01011 \\
0111010101 & 0111010101 & 0030001110 & 10101 \\
0011111010 & 0011111010 & 0003000111 & 11010 \\
1001101101 & 1001101101 & 0000310011 & 01101 \\
0100100110 & 0100101001 & 1100130110 & 11111 \\
1010000011 & 1010010100 & 1110003011 & 11111 \\
0101010001 & 0101001010 & 0111010301 & 11111 \\
0010111000 & 0010100101 & 0011111030 & 11111 \\
1001001100 & 1001010010 & 1001101103 & 11111 \\
0100100110 & 0011000110 & 1011011111 & 31001 \\
1010000011 & 0001100011 & 0101111111 & 13100 \\
0101010001 & 1000110001 & 1010111111 & 01310 \\
0010111000 & 1100011000 & 1101011111 & 00131 \\
1001001100 & 0110001100 & 0110111111 & 10013
\end{pmatrix}$$

Fig 3: Matrix-3

$$
\begin{pmatrix}
1\ \ 1\ \ 1\ 11\ 1\ 11111\ \ 1\ 1\ \ 1\ \ \ 11\ \ 1\ \ \ 111\ \ \ 111\ \ \ 1\ \ \ 11\ \ 11\ \ \ 1\ \ 11\ \ \ 11\ \ \ 1\ \ \ 11 \\
1-1-111-111111\ -11-1-11-1-111-111-1-11-11-1-11-11-1-11 \\
1-1-1-111111111\ -11-1-1-1-1-111111\ -1-11-11-1-11-11-1-1 \\
11-1-1-1111111\ -11-11-11-1-1-11-1111-1-11-11-1-11-11-1 \\
111-1-1-111111\ -1-11-1111-1-1-1-1-1111-1-11-11-1-11-11 \\
1-111-1-1111111\ -1-11-1-111-1-11-1-1111-1-11-11-1-11-1 \\
111111\ -11-1-11-11-1-11-1-111-11-111-1-1-111-1-1-111-1 \\
1111111\ -11-1-11-11-1-1-1-1-111-11-111-1-1-111-1-1-111 \\
111111\ -11-11-1-11-11-11-1-1-111-11-111-1-1-111-1-1-11 \\
111111\ -1-11-11-1-11-1111-1-1-111-11-111-1-1-111-1-1-1 \\
1111111\ -1-11-11-1-11-1-111-1-1-111-11-111-1-1-111-1-1 \\
1-11-1-11-11-1-11-1-111-11111111\ -1-11-11-1-11-1-111-1 \\
11-11-1-11-11-1-1-1-1-1111111111\ 1-1-11-11-1-1-1-1-111 \\
1-11-11-1-11-11-11-1-1-1111111\ -1111-1-11-11-11-1-1-11 \\
1-1-11-11-1-11-1111-1-1-111111\ -1-1111-1-11-1111-1-1-1 \\
11-1-11-11-1-11-1-111-1-1111111\ -1-1111-1-11-1-111-1-1 \\
1-1-111-1-1-111-111111\ -11-1-111-111-1-11-1-11-1-111-1 \\
1-1-1-111-1-1-111111111\ -11-1-1-11-1111-11-1-1-1-1-111 \\
11-1-1-111-1-1-1111111\ -11-11-11-11-11-11-11-11-1-1-11 \\
111-1-1-111-1-1-111111\ -1-11-1111-11-1-1-11-1111-1-1-1 \\
1-111-1-1-111-1-1111111\ -1-11-1-111-111-1-11-1-111-1-1 \\
111-1-111-111-111-1-111-111-1-1-1-1-1-111-1-111-111-1 \\
1111-1-1-11-111111\ -1-1-11-111-1-1-1-1-1111-1-1-11-111 \\
1-1111-11-11-11-1111-11-11-11-1-1-1-1-1-1111-11-11-11 \\
1-1-111111\ -11-1-1-111111\ -11-1-1-1-1-1-1-1-111111\ -11-1 \\
11-1-111-111-111-1-111-111-11-1-1-1-1-11-1-111-111-11 \\
1-11-1-11-1-111-1-11-1-11-11-1-1111-1-11-1-111-111111 \\
11-11-1-1-1-1-1111-11-1-11-11-1-1111-1-1-1-1-11111111 \\
1-11-11-11-1-1-11-11-11-1-11-11-1-1111-11-1-1-1111111 \\
1-1-11-1111-1-1-1-1-11-11-1-11-11-1-111111\ -1-1-111111 \\
11-1-11-1-111-1-11-1-11-11-1-11-11-1-111-111-1-111111 \\
1-11-1-11-1-111-1-1-111-1-1-111-11-111-111111\ -11-1-11 \\
11-11-1-1-1-1-111-1-1-111-1-1-111-11-111111111\ -11-1-1 \\
1-11-11-11-1-1-111-1-1-111-1-1-111-11-1111111\ -11-11-1 \\
1-1-11-1111-1-1-111-1-1-111-1-1-111-11-111111\ -1-11-11 \\
11-1-11-1-111-1-1-111-1-1-111-1-1-111-11111111\ -1-11-1
\end{pmatrix}
$$

Fig 4: Matrix-4

$$
\begin{pmatrix}
i-11-1-11111\ -1-11-1-1111111\ -1-111-11 \\
-1i-11-1-11111\ -1-111-1111111\ -1-111-1 \\
1-1i-11-1-11111\ -1-1-11-1111111\ -1-111 \\
-11-1i-11-1-11111\ -11-11-1111111\ -1-11 \\
-1-11-1i-11-1-1111111\ -11-1111111\ -1-1 \\
1-1-11-1i-11-1-1111-111-11-1111111\ -1 \\
11-1-11-1i-11-1-111-1-111-11-1111111 \\
111-1-11-1i-11-1-111-1-111-11-111111 \\
1111-1-11-1i-11-1-111-1-111-11-11111 \\
-11111\ -1-11-1i-11-1111-1-111-11-1111 \\
-1-11111\ -1-11-1i-111111\ -1-111-11-111 \\
1-1-11111\ -1-11-1i-111111\ -1-111-11-11 \\
-11-1-11111\ -1-11-1i111111\ -1-111-11-1 \\
-11-111-1-1111111\ i1-111-1-1-1-111-11 \\
1-11-111-1-1111111\ i1-111-1-1-1-111-1 \\
11-11-111-1-11111\ -11i1-111-1-1-1-111 \\
111-11-111-1-11111\ -11i1-111-1-1-1-11 \\
1111-11-111-1-11111\ -11i1-111-1-1-1-1 \\
11111\ -11-111-1-11-111-11i1-111-1-1-1 \\
111111\ -11-111-1-1-1-111-11i1-111-1-1 \\
-1111111\ -11-111-1-1-1-111-11i1-111-1 \\
-1-1111111\ -11-111-1-1-1-111-11i1-111 \\
1-1-1111111\ -11-111-1-1-1-111-11i1-11 \\
11-1-1111111\ -11-111-1-1-1-111-11i1-1 \\
-111-1-1111111\ -11-111-1-1-1-111-11i1 \\
1-111-1-1111111\ -11-111-1-1-1-111-11i
\end{pmatrix}
$$

Fig 5: Matrix-5

$$\left[\begin{array}{l}
i1-1-11111-11-11-1-1-1-1-1-1-1111-11111-11111-11-1-1-1111111-1-11-11-11 \\
1i1-1-11111-1-1-11-1-11-1-1-1-1111-11111-11-111-111-1-111-1111-111-11-1 \\
-11i1-1-11111-1-1-11-1-11-1-1-11111-11111-11-111-111-1-11-1-1111-111-11 \\
-1-11i11-1111-1-1-1-11-1-11-1-1-11111-11111-11-111111-1-11-1-1111-111-1 \\
1-1-11i11-1111-1-1-1-1-1-1-11-11-11111-11111-11-11-1111-111-1-11-11-111 \\
11-111i1-1-11111-11-11-1-1-1-1-1-1-111-11-1111-11111-11-1-1-1111111-1-1 \\
111-111i1-1-11111-1-1-11-1-11-1-1-1-111-11-1111-11-111-111-1-111-1111-1 \\
1111-1-11i1-1-11111-1-1-11-1-11-1-1-1-111-111111-11-111-111-1-11-1-1111 \\
-11111-1-11i11-1111-1-1-1-11-1-11-1-11-111-1-11111-11-111111-1-11-1-111 \\
1-11111-1-11i11-1111-1-1-1-1-1-1-11-1-11-1111-11111-11-11-1111-111-1-11 \\
-1-1-1-1111-111i1-1-11111-11-11-1-1-1111-1-11-11-1111-11111-11-1-1-1111 \\
1-1-1-1-1111-111i1-1-11111-1-1-11-1-1-1111-111-11-1111-11-111-111-1-111 \\
-11-1-1-11111-1-11i1-1-11111-1-1-11-1-1-1111-111-111111-11-111-111-1-11 \\
-1-11-1-1-11111-1-11i11-1111-1-1-1-111-1-1111-111-1-11111-11-111111-1-1 \\
-1-1-11-11-11111-1-11i11-1111-1-1-1-111-1-11-11-1111-11111-11-11-1111-1 \\
-11-1-1-1-1-1-1-1111-111i1-1-11111-11-1-1111111-1-11-11-1111-11111-11-1 \\
-1-11-1-11-1-1-1-1111-111i1-1-11111-11-1-111-1111-111-11-1111-11-111-11 \\
-1-1-11-1-11-1-1-11111-1-11i1-1-1111111-1-11-1-1111-111-111111-11-111-1 \\
-1-1-1-11-1-11-1-1-11111-1-11i11-1111111-1-11-1-1111-111-1-11111-11-111 \\
1-1-1-1-1-1-1-11-11-11111-1-11i11-111-1111-111-1-11-11-1111-11111-11-11 \\
111-11-11-1-1-1-1-1-1-1111-111i1-1-1111-11-1-1-1111111-1-11-11-1111-111 \\
1111-1-1-11-1-11-1-1-1-1111-111i1-1-1-111-111-1-111-1111-111-11-1111-11 \\
-11111-1-1-11-1-11-1-1-11111-1-11i1-11-111-111-1-11-1-1111-111-111111-1 \\
1-1111-1-1-1-11-1-11-1-1-11111-1-11i1-11-111111-1-11-1-1111-111-1-11111 \\
11-1111-1-1-1-1-1-1-11-11-11111-1-11i1-11-11-1111-111-1-11-11-1111-1111
\end{array}\right.$$