Analysis Of M/M/1 Queueing Model With Applications To Waiting Time Of Customers In Banks.

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***Abstract*-In this paper, an attempt is made to review the analysis of Stochastic Birth-Death Markov processes which turns out to be a highly suitable modeling tool for many queueing systems in general and M/M/1 queueing model in particular. The model, M/M/1, as a single-channel queueing system with Poisson arrivals and exponential service and with queueing discipline of first come first serve basis, is applied to arrivals and waiting times of customers in Intercontinental Bank PLC, Ile-Ife Branch, Osun State, Nigeria. The queue size of customers including traffic intensity and average number of customers in the system and queue; the service/waiting times of customers including the average time spent in the system and queue by a customer, are all obtained. The traffic intensity obtained is 0.8378 which indicate the probability of a customer queueing or waiting for service on arrival.**

Keywords- stochastic birth-death Markov process, Chapman-Kolmogorov equations, Poisson arrivals, exponential service rates, traffic intensity.

I. NTRODUCTION

1) *Background Information to the Study*

The success of economic policies aiming at economic growth and development of any nation in the world today can only be adequately achieved under a sound banking system. A sound banking system motivates and invites investors from within and outside the country. On the other hand, more efficient and effective investments are basis for a good foreign exchange that in turn generates more external reserves to boost a nation's economy. Any nation with insufficient foreign reserves is bound to borrow from richer and more developed nations to cater for her citizens.Banks can affect liquidity in an economy and they can also directly influence the rate of growth of investments into certain sectors of the economy like agric sector, industrial sector, housing sector, education sector, to mention but a few. Banks invest customers' funds, keep deposits and also give out loans to customers.The recent sweeping measures by the Central Bank of Nigeria with the full support of the Federal Government of Nigeria to sanitize banking sector in the country and the injection of about 620 billion Naira into the industry which have complement the CBN's earlier recapitalization policy are all good examples of the importance of the banking sector to the nation's economy. These measures are presently and gradually reducing the earlier problems of liquidation and distress in this sector.

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2) *Statement of the Problems*

In spite of all the efforts of the concerned stake holders in Nigeria most especially the Federal Government and the Central Bank of Nigeria (CBN) to improve the services and effective performance of banks in the country, there are still some avoidable problems that are militating against the success of this sector. One of the most frequent of them is the problem of waiting lines (queues) found in virtually all the banks in the country. Queue is a very volatile situation which always cause unnecessary delay and reduce the service effectiveness of establishments. Long queueing in banks has many negative effects to customers and even the affected banks as well. These negative effects include wasting of man-hours, chaos, unnecessary congestion in banking halls which may lead to suffocation and contraction of communicable diseases. Above all, cases have been witnessed where customers, while waiting too long in banks to lodge in or cash some money got bombarded by armed robbers who killed, mimed, injured both customers and banking officials and get away with huge amount of money from both the customers and the affected banks at large. Also an ill-health or aged customer, while waiting too long on a queue without being attended to in good time, may develop complications, faint or even slum; this may lead to death if proper and urgent medical attention is not provided.
Application of the theory of queue as the development of mathematical models to study and analyze waiting lines with the hope of reducing this social phenomenon in our banking systems would go a long way in improving their services. Through better understanding of queuing situation and the application of appropriate models to deal with it, the bank management will be able to make and take good decisions that will be in the best interest of the overall customers and other stake holders. The end result would be high-volume of customers which will in turn bring more cash, profits, investments, shareholders and the like which will directly or indirectly improve our economy. It is against this background that this research is proposed with the research objectives in the next section on the case study– Intercontinental Bank PLC, Ile-Ife Branch (since the queue problems that are being witnessed in this bank is very much

similar to what is encountered in all banks across the country).

3) *Research Objectives*

Applying M/M/1 Queueing Model to waiting lines (queues) in Intercontinental Bank PLC, Ile-Ife Branch, we want to:

i. To determine the arrivals and service rates per unit time of customers.
ii. To determine the queue size of customers including traffic intensity,expected number of customers in the system and in the queue.
iii. To estimate the waiting time of customers which include the expected time that a customer will spend in the system and in the queue.

4) *Source Of Data For The Study*

Primary data on arrivals, waiting time, service patterns and departures of customers in Intercontinental Bank PLC, Ile-Ife, Osun State was collected for upwards of a month (i.e. 21 working days) and used as input data into the model (M/M/1) to obtain: the queue size of customers including traffic intensity and average number of customers in the system and in the queue, the waiting time of customers including the average time that a customer will spend in the system and in the queue.

5) *Research Methodology*

The research method will involve derivation of various M/M/m queueing systems generally through Birth-Death Markov Processes and M/M/1 Queuing Model in particular which always exhibits Markov behaviour conformity, as such makes such queuing system analytically(mathematically) tractable.Furthermore, an analysis and interpretation of results obtained, applying M/M/1queueing model, will be done using primary data collected which include arrivals, waiting time, service patterns and departures of customers in Intercontinental Bank PLC, Ile-Ife Branch, Osun State, Nigeria.

II. Model Specification

1) *Birth –Death Markov Processes*

Stochastic birth-death Markov processes turns out to be a highly suitable modeling tool for many queueing processes. Examples of these are any M/M/n queueing models which are our preoccupation in this study with much emphasis on M/M/1 model. Full examination of these models will be considered later in this section.Let $\{N(t), t \geq 0\}$, with parameter set T, be an integer-valued continuous-time discrete stochastic process. The discrete state space of the process comprises (or can assume) non-negative integer values $0, 1, 2, \dots, \infty$. Here $N(t)$ is interpreted as the random number of members in some population as a function of time. In other words $N(t)$ is viewed as the size of population at time t.By assumption, the classical Markov property is imposed as a restriction on the process $N(t)$, i.e. given the value of $N(s)$, the values of $N(s+t) for\ t > 0$ are not influenced by the values of $N(u)$ for $u < s$. In words, the way in which the entire past history affects the future of the process is completely summarized in the current state of the process, In other words; only the present state gives any information of the future behavior of the process, knowledge of the history of the process does not add new information (The future development of a continuous-time Markov process depends only on its present state and not on its evolution in the past). Expressed analytically the Markov property may be defined as thus: A stochastic process $\{N(t), t \geq 0\}$ with set T and discrete state space

is called a continuous-time Markov process (chain) if for any $m \geq 1$

$$P[N(t_{m+1}) = n_{m+1}/N(t_m) = n_m, \dots N(t_1) = n_1] = P[N(t_{m+1}) = n_{m+1}/N(t_m) = n_m] \qquad (1)$$

And it should be valid for all $t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1}\ and\ any\ m$

2) *Transition Probabilities*

In equation (1), set $t_m = s,\ n_m = i,\ t_{m+1} = s + t\ and\ n_{m+1} = j$. then the right-hand side of the equation expresses the probability that the process makes a transition from state i at time s to state j in time t relative to s. Such a probability, denoted by $p_{i,j}(s,t)$, is referred to as a state transition probability for the Markov process. In this research work, we are only concerned with transition probabilities being independent of absolute time$s$, i.e. for all $s > 0$ we have:

$$p_{i,j}(s,t) = p_{i,j}(t) = P\left[N(t) = \frac{j}{N(0)} = i\right] = P\left[N(s+t) = jNs = i\right] \qquad (2)$$

This is called time-homogeneous or stationary transition probabilities. In other words, the intensity of leaving a state is constant in time. It is always natural to make the following definition (for the sake of clarity) which states that the transition probabilities only depends on which state the process is in and not on the time.

Definition: Let $\{N(t),\ t \geq 0\}$ be a discrete Markov process. If the conditional probabilities $P[N(s+t) = j/N(s) = i], for\ s,t \geq 0,$ do not depend on s, the process is said to be time homogeneous. Then we define the transition probability $p_{i,j}(t) = P[N(t) = j/N(0) = i]$ and the transition matrix **P(t)**, whose element with index $(i,j)\ is\ p_{i,j}(t)$

It is generally assumed that the transition probabilities $p_{i,j}(t)$ are well behaved in the sense that they are all continuous and the derivative exists.

Note that $p_{ii}(0) = 1\ and\ p_{ij} = 0\ for\ i \neq j\ so\ that$ $\mathbf{P(t) = I}$

For a Markov process with time-homogeneous transition probabilities the so called Chapman-Kolmogorov equation implies

$$p_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s) \qquad (3)$$

This equation states that in order to move from state i to j in time $(t+s)$, the queue size process $N(t)$ moves to some intermediate state $\boldsymbol{k}$ in time $\boldsymbol{t}$ and then from $\boldsymbol{k}$ to $\boldsymbol{j}$ in the remaining time $\boldsymbol{s}$. It also says how to compute the long - interval transition probability from a sum of short – interval transition probability components.

An infinitesimal transition probability, denoted by $\boldsymbol{p_{ij}}(\Delta \boldsymbol{t})$, specifies the *immediate* probabilistic behavior of a Markov process in that $(\Delta \boldsymbol{t}) \to \boldsymbol{0}$. By help of equation (63) it turns out that any transition probability $\boldsymbol{p_{ij}}(\boldsymbol{t})$ can in principle be determined if the infinitesimal transition probabilities are known. Hence, the overall probabilistic behaviour of a Markov process is ultimately given by the infinitesimal transition probabilities. Together they define the transition kernel of the process.

III. Generalized Markov Birth-Death Process

Suppose now that the population size changes by births and deaths. A birth-death Markov process is characterized by the fact that the discrete state variable changes by at most one, if it changes at all, during an infinitely small time interval. The generalized Markov birth-death process thus satisfies the following criteria:

1. The probability distributions governing the numbers of births and deaths in a specific time interval depends on the length of the interval but not on its starting point
2. The probability of exactly one birth in a small time interval,$\Delta \boldsymbol{t}$ given that the population size at time $\boldsymbol{t}$ is $\boldsymbol{n}$ is $\boldsymbol{\lambda_n}\Delta \boldsymbol{t} + \boldsymbol{0}(\Delta \boldsymbol{t})$, where $\boldsymbol{\lambda_n}$ is a constant,
3. The probability of one death in small time interval $\Delta \boldsymbol{t}$ given that the population size at time $\boldsymbol{t}$ is $\boldsymbol{n}$ is $\boldsymbol{\mu_n}\Delta \boldsymbol{t} + \boldsymbol{0}(\Delta \boldsymbol{t})$ where $\boldsymbol{\mu_n}$ is a constant.
4. The probability of more than one birth and the probability of more than one death in a small time interval $\Delta \boldsymbol{t}$ are both $\boldsymbol{0}(\Delta \boldsymbol{t})$.

Reflecting these facts, the following postulations specify the transition kernel of a general birth-death Markov process:

$$\boldsymbol{P}[\boldsymbol{N}(\boldsymbol{t}+\Delta \boldsymbol{t}) = \boldsymbol{n}+\boldsymbol{1}/\boldsymbol{N}(\boldsymbol{t}) = \boldsymbol{n}] = \boldsymbol{\lambda_n}\Delta \boldsymbol{t} + \boldsymbol{0}(\Delta \boldsymbol{t}), \qquad \boldsymbol{n} \geq \boldsymbol{0}$$
$$\boldsymbol{P}[\boldsymbol{N}(\boldsymbol{t}+\Delta \boldsymbol{t}) = \boldsymbol{n}-\boldsymbol{1}/\boldsymbol{N}(\boldsymbol{t}) = \boldsymbol{n}] = \boldsymbol{\mu_n}\Delta \boldsymbol{t} + \boldsymbol{0}(\Delta \boldsymbol{t}), \qquad \boldsymbol{n} \geq \boldsymbol{1}$$
$$\boldsymbol{P}[\boldsymbol{N}(\boldsymbol{t}+\Delta \boldsymbol{t}) = \boldsymbol{n}/\boldsymbol{N}(\boldsymbol{t}) = \boldsymbol{n}] = \boldsymbol{1} - (\boldsymbol{\lambda_n}+\boldsymbol{\mu_n})\Delta \boldsymbol{t} + \boldsymbol{0}(\Delta \boldsymbol{t}), \qquad \boldsymbol{n} \geq \boldsymbol{1}$$
$$\boldsymbol{P}[\boldsymbol{N}(\boldsymbol{t}+\Delta \boldsymbol{t}) = \boldsymbol{k}/\boldsymbol{N}(\boldsymbol{t}) = \boldsymbol{n}] = \boldsymbol{0}(\Delta \boldsymbol{t}), \qquad |\boldsymbol{k}-\boldsymbol{n}| \geq \boldsymbol{2} \qquad \textbf{(4)}$$

Here $\boldsymbol{0}(\Delta \boldsymbol{t})$ is a quantity such that $\lim_{\Delta t \to \infty} \boldsymbol{0}(\Delta_t) = \boldsymbol{0}$. The first equation handles the case when the state variable increases by one i.e. $\boldsymbol{N}(\boldsymbol{t}) = \boldsymbol{n}+\boldsymbol{1}$. This is referred to as single birth. Here $\boldsymbol{\lambda_n}$ is proportionality constant such that the product $\boldsymbol{\lambda_n \Delta t}$ should reflect the probability for a single birth to happen during the infinitesimal time interval $(\boldsymbol{t}, \boldsymbol{t}+\boldsymbol{\Delta t})$. It is customary to interpret $\boldsymbol{\lambda_n}$ as the instantaneous birth rate. Likewise, the second equation is for the case when the state variable is reduced by one i.e. $\boldsymbol{N}(\boldsymbol{t}) = \boldsymbol{n}-\boldsymbol{1}$. This is referred to as single death. The product $\boldsymbol{\mu_n}\Delta \boldsymbol{t}$ signifies the probability that a single death takes place. $\boldsymbol{\mu_n}$ denotes the instantaneous death rate. The third equation handles the case when the state variable does not change i.e. $\boldsymbol{1} - (\boldsymbol{\lambda_n}+\boldsymbol{\mu_n})\boldsymbol{\Delta t}$ reflects the probability that neither a single birth nor single death occur, i.e. $\boldsymbol{N}(\boldsymbol{t}) = \boldsymbol{n}$, during the infinitely small time interval. Multiple births, multiple deaths and simultaneous births and deaths are taken care of by the $\boldsymbol{0}(\boldsymbol{\Delta t})$ terms in the equations. This should be interpreted such that the probability for these events to happen is negligible as $\boldsymbol{\Delta t} \to \boldsymbol{0}$, we say that multiple events are prohibited.

We should note that the transition probabilities from (4) are in general state dependent. This is so since the instantaneous births rate $\boldsymbol{\lambda_n}$ and also the death rate $\boldsymbol{\mu_n}$ may depend on the departing state $\boldsymbol{n}$. A small comment also applies to the second and third equations. Since no death can occur if the state variable is already zero i.e. if $\boldsymbol{n} = \boldsymbol{0}$, we always define $\boldsymbol{\mu_0} = \boldsymbol{0}$.

By combining equation (3) with the infinitesimal transition probabilities from (4) and using the notation $\boldsymbol{P_n}(\boldsymbol{t}) = \boldsymbol{P}[\boldsymbol{N}(\boldsymbol{t}) = \boldsymbol{n}]$, we can write:

$$\boldsymbol{P_n}(\boldsymbol{t}+\boldsymbol{\Delta t}) = \sum_{k=0}^{\infty} \boldsymbol{P}[\boldsymbol{N}(\boldsymbol{t}+\boldsymbol{\Delta t}) = \boldsymbol{n}/\boldsymbol{N}(\boldsymbol{t}) = \boldsymbol{k}]\boldsymbol{P}[\boldsymbol{N}(\boldsymbol{t})\boldsymbol{k}] \qquad (\boldsymbol{Bayes'formular})$$
$$= \boldsymbol{P_n}(\boldsymbol{t})[\boldsymbol{1} - (\boldsymbol{\lambda_n}+\boldsymbol{\mu_n})\Delta \boldsymbol{t} + \boldsymbol{0}(\Delta \boldsymbol{t})] + \boldsymbol{P_{n-1}}(\boldsymbol{t})[\boldsymbol{\lambda_{n-1}}(\Delta \boldsymbol{t}) + \boldsymbol{0}(\Delta \boldsymbol{t}) + \boldsymbol{P_{n+1}}(\boldsymbol{t})[\boldsymbol{\mu_{n+1}}(\Delta \boldsymbol{t}) + \boldsymbol{0}(\Delta \boldsymbol{t})] + \boldsymbol{0}(\Delta \boldsymbol{t}) \qquad \textbf{(5)}$$

By rearranging terms and dividing by $\Delta \boldsymbol{t}$, we have

$$\frac{\boldsymbol{P_n}(\boldsymbol{t}+\Delta \boldsymbol{t}) - \boldsymbol{P_n}(\boldsymbol{t})}{\Delta \boldsymbol{t}} = -(\boldsymbol{\lambda_n}+\boldsymbol{\mu_n})\boldsymbol{P_n}(\boldsymbol{t}) + \boldsymbol{\mu_{n+1}}\boldsymbol{P_{n+1}}(\boldsymbol{t}) + \boldsymbol{\lambda_{n-1}}\boldsymbol{P_{n-1}}(\boldsymbol{t}) \qquad \textbf{(6)}$$

Taking limit as $\Delta \boldsymbol{t} \to \boldsymbol{0}$, we obtain the differential equation:

$$\boldsymbol{P'_n}(\boldsymbol{t}) = \frac{\boldsymbol{dP_n}(\boldsymbol{t})}{\boldsymbol{dt}} = -(\boldsymbol{\lambda_n}+\boldsymbol{\mu_n})\boldsymbol{P_n}(\boldsymbol{t}) + \boldsymbol{\mu_{n+1}}\boldsymbol{P_{n+1}}(\boldsymbol{t}) + \boldsymbol{\lambda_{n-1}}\boldsymbol{P_{n-1}}(\boldsymbol{t}), \quad \boldsymbol{n} = \boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}, \ldots \qquad \textbf{(7)}$$

While (7) holds for $\boldsymbol{n} = \boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}, \ldots$, we also require an equation for $\boldsymbol{n} = \boldsymbol{0}$. By following a logical argument as above, we can write

$$\frac{\boldsymbol{dP_n}(\boldsymbol{t})}{\boldsymbol{dt}} = -\boldsymbol{\lambda_0}\boldsymbol{P_0}(\boldsymbol{t}) + \boldsymbol{\mu_1}\boldsymbol{P_1}(\boldsymbol{t}) \qquad (8)$$

Equation (7) is the general model equation for a birth-death Markov process and it essentially captures the probabilistic dynamics of the process. The equation is a differential equation in the continuous time variable $\boldsymbol{t}$ and a difference equation (also called a recurrence equation).

As already shown, note that the model equation (7) is valid for $\boldsymbol{t} > 0 \; and \; n = 0,1,2,\ldots,\infty$. For $\boldsymbol{t} = \boldsymbol{0}$ we have a boundary condition and it is customary to define $\boldsymbol{P_n}(\boldsymbol{0}) = \boldsymbol{\delta_{ij}} = \boldsymbol{1} \; \boldsymbol{for} \; \boldsymbol{i}$

$$\boldsymbol{and} = \boldsymbol{0} \; \boldsymbol{otherwise} \qquad (9)$$

Hence, in zero time the process will certainly not move.

3) *Steady State Solution*

We now examine the process when it is in equilibrium (steady state). Under proper conditions, such equilibrium will be reached after the system has been operating for some time. Equilibrium in turn, implies that the state probabilities $\boldsymbol{P_n}(\boldsymbol{t})$ eventually become independent of $\boldsymbol{t}$ and approach a

set of constant values (if it exists) which is denoted by $P_n,\ n = 0, 1, 2 \ldots$ as $t \to \infty$ where $P_n = \lim_{t\to\infty} P_n(t)$. This can be interpreted as steady state probability that there are n users in the system. Also under these circumstances in the steady state, $\lim_{t\to\infty} P'_n(t) = 0$.
Given the above, (7) and (8) are then transformed to:

$$0 = -(\lambda_n + \mu_n)P_n + \mu_{n+1}P_{n+1}\lambda_{n-1}P_{n-1},\quad n = 1, 2, 3, \ldots$$

$$or\ \ (\lambda_n + \mu_n)P_n = \mu_{n+1}P_{n+1} + \lambda_{n-1}P_{n-1} \tag{10}$$

And $0 = -\lambda_0 P_0 + \mu_1 P_1$

$$or\ \ \lambda_0 P_0 = \mu_1 P_1 \tag{11}$$

Equations (10) and (11) are called the equilibrium – equations or balance equations of birth and death Markov process. They have a natural and useful interpretation. They say that, at equilibrium,

IV. THE PROBABILITY FLOW OUT OF A STATE = THE

1) *probability of flow in that state*

In other words, for any time t when the system is in equilibrium, the probability of observing a transition out of state n in the next Δt must be equal to the probability of observing a transition into state n This key observation will allow us in most cases to generate the correct equilibrium equations without going through the burden of writing down equations like (5).
For the process, another set of balance equations, even easier than (10) and (11) can be obtained directly from (11) i.e.

$$\lambda_0 P_0 = \mu_1 P_1$$
$$\lambda_1 P_1 = \mu_2 P_2$$
$$\lambda_2 P_2 = \mu_3 P_3$$
$$. \qquad .$$

and in general, $\lambda_n P_n = \mu_{n+1} P_{n+1}$

$$for\ n = 0, 1, 2, \ldots \tag{12}$$

We can now proceed to solve the balance equation expressing all steady-state probabilities $P_n, n = 0, 1, 2, \ldots$ in terms of one of them and then taking advantage of the fact that

$$\sum_{n=0}^{\infty} P_n = 1 \tag{13}$$

this is called the normalization equation.
It is common practice to express $P_1, P_2, P_3, \ldots$ in terms of P_0, the steady – state probability of an empty system. Working with (10) and (11) or equivalently (and preferably) with (12) we have (solving recursively):

$$P_1 = \frac{\lambda_0}{\mu_0} P_0 \tag{14}$$

$$P_2 = \frac{\lambda_1}{\mu_2} P_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0 \tag{15}$$

and, in general

$$P_n = \frac{\lambda_{n-1}\lambda_{n-2} \ldots\ldots\ldots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \ldots\ldots \mu_2 \mu_1} P_0 \tag{16}$$

or, defining the coefficient of P_0, in (16) as the quantity K_n, we have

$$P_n = K_n P_0 \qquad for\ n = 1, 2, 3, \ldots$$

Going back to (6.13)

$$\sum_{n=0}^{\infty} P_n = (1 + \sum_{n=1}^{\infty} K_n) P_0 \tag{17}$$

Or $$P_0 = \frac{1}{(1 + \sum_{n=1}^{\infty} K_n)} \tag{18}$$

It follows that the system can reach steady-state (or equilibrium distribution exists) if and only if $\sum_{n=1}^{\infty} K_n) < \infty$. For, otherwise, $P_0 = P_1 = P_2 = P_3 = \ldots = 0$ (i.e. the number of users in the system never "stabilizes")

2) *Analysis of the M/M/1 Queueing Model*

In this queueing system, the customers arrive according to Poisson process. Inter-arrival times are exponentially distributed with average arrival rate λ. The service times (i.e. the time it takes to serve every customer) are exponentially distributed with average service rate μ. The service times are mutually independent and further independent of the inter-arrival times. The constant average arrival rate λ and the average service rate μ are in units of customers per unit time. The expected inter-arrival time and the expected service time are $1/\lambda$ and $1/\mu$ respectively.
There is only one server; no limit on the system capacity, i.e. the buffer is assumed to be infinite and the queueing discipline is first-come-first served (FCFS).
Since exponentially distributed inter-arrival times with mean $1/\lambda$ are equivalent, over a time interval, say τ, to a Poisson – distributed arrival pattern with mean $\lambda\tau$, M/M/1 queueing model is often refer to as single-server, infinite-capacity queueing systems with customers arriving according to Poisson process and exponential service times. When a customer enters an empty system, his service starts at once; if the system is nonempty the incoming customer joins the queue. When a service completion occurs, a
customer from the queue, if any, enters the service facility at once to get served.M/M/1 queueing model is a Poisson birth - death process. A birth occurs when a customer arrives and a death occurs when a customer departs. Both processes are modeled as memoryless Markov process. The M designation in M/M/1 actually refers to this memoryless/Markov feature of the arrival and service processes (i.e. they are exponentially distributed).The memoryless property can be further described as thus: with exponential inter-arrival times and exponential service times, the distribution of the time until the next arrival and/or service completion is not affected by the time that elapsed since the last arrival and the last completion.
System State: Due to the memoryless property of the exponential distribution, the entire state of the system, as far as the concern of probabilistic analysis, can be summarized by the number of customers in the system, n (i.e. the customers waiting in the queue and the one being served). – the past/history (how we get there) does not matter. When a customer arrives or departs, the system moves to an adjacent state
(either $n + 1$ or $n - 1$.) Let $N(t)$ be the number of customers in the system at time t.
Theorem:

The process $\{N(t), t \geq 0\}$ is a birth and death process with birth rate

$\lambda_n = \lambda \; for\; all\; n \geq 0$ and with death rate $\mu_n = \mu \; for\; all\; n \geq 1$.

Proof:

Because of the exponential distribution of the inter-arrival times and of the service times, it should be clear that $\{N(t), t \geq 0\}$ is a Markov process. On the other hand, since the probability of having two events (departures, arrivals) in the interval of time $(t, t+\Delta t)$ is $0(\Delta t)$, which is characterized by the fact that the discrete state variable changes by at most one, if it changes at all, during an infinitely small time interval (criteria for the generalized Markov birth-death process), we have the following transitional probabilities:

$$P[N(t+\Delta t) = n+1/N(t) = n] = \lambda\Delta t + 0(\Delta t), \quad n \geq 0$$
$$P[N(t+\Delta t) = n-1/N(t) = n] = \mu\Delta t + 0(\Delta t), \quad n \geq 1$$
$$P[N(t+\Delta t) = n/N(t) = n] = 1-(\lambda+\mu)\Delta t + 0(\Delta t), \quad n \geq 1$$
$$P[N(t+\Delta t) = n/N(t) = n] = 1-\lambda\Delta t + 0(\Delta t), \quad n = 0$$
$$P[N(t+\Delta t) = k/N(t) = n] = 0(\Delta t), \quad |k-n| \geq 2. \qquad (19)$$

Here $0(\Delta t)$ is a quantity such that $\lim_{\Delta t \to \infty} 0(\Delta_t) = 0$.

This shows that $\{N(t), t \geq 0\}$ is a birth and death process.

From (6), (7), and (8) with the fact that $\lambda_n = \lambda$ and $\mu_n = \mu \; for\; all\; n$ in the birth and death Markov process, the differential equations for the transitional probability that the system is in state n at time t (for M/M/1 system) is given as:

$$P'_n(t) = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t), \quad n = 1, 2, 3, \ldots \qquad (20)$$
$$P'_n(t) = -\lambda P_0(t) + \mu P_1(t), \quad n = 0 \qquad (21)$$

As $t \to \infty$, $P'_n(t) \to 0 \; and \; P_n(t) \to P_n$

Thus we have:

$$0 = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1}, \quad n = 1, 2, 3, \ldots$$

Or $(\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+1}$ (22)

And $0 = -\lambda P_0 + \mu P_1, \quad n = 0$

Or $\lambda P_0 = \mu P_1$ (23)

Which are the equilibrium or balance equations for an M/M/1 system

Solving recursively (using (7.3)), we have

$$\lambda P_0 = \mu P_1$$
$$\lambda P_1 = \mu P_2$$
$$\lambda P_2 \qquad (24)$$
$$. \qquad .$$

We define $\rho = \lambda/\mu$ (25)

The quantity ρ is referred to as the traffic intensity (or the utilization factor) since it gives the mean quantity of work brought to the system per unit of time.

Now, from (24),

$$P_1 = \lambda/\mu P_0 = \rho P_0$$
$$P_2 = \lambda/\mu P_1 = \rho(\rho P_0) = \rho^2 P_0$$
$$P_3 = \lambda/\mu P_2 = \rho(\rho^2 P_0) = \rho^3 P_0$$

In general $P_n = \rho^n P_0$ (26)

Since the sum of the probabilities must equal unity

i.e. $\sum_{n=0}^{\infty} P_n = 1 \; and \; 0 < \rho < 1$

i.e. $\sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} \rho^n P_0 = 1$

we have $P_0 \sum_{n=0}^{\infty} \rho^n = 1$, $\left(\frac{1}{1-\rho}\right) P_0 = 1$

$$\therefore P_0 = 1-\rho \quad or \quad 1-\lambda/\mu \qquad (27)$$

That is

$$P_n = \rho^n(1-\rho), \quad n > 0, \rho < 1 \qquad (28)$$

This is called equilibrium distribution (or queue-length of density function of an M/M/1 queue)

The stability condition, $\rho < 1$ simply says that the system is stable if the work that is brought to the system per unit time is strictly smaller than the processing rate (which is 1 here since there is only one server).

It is noteworthy that the queue will be empty infinitely many times when the system is table i.e. when $n = 0$; from (28)

$$P_0 = 1-\rho > 0 \qquad (29)$$

We observe that the result in (28) which is the density function of the queue-length in a steady-state is a geometric distribution. We then compute (in particular) the mean number of customers as:

$$E(N) = \frac{\rho}{1-\rho} \qquad (30)$$

We can observe that $E(N) \to \infty \; as \; \rho \to 1$ so that in practice, if the system is not stable, then the queue will explode. We also find from (28) that:

$$V(N) = \frac{\rho}{(1-\rho)^2} \qquad (31)$$

Again, we find from (28) that the probability that the queue exceeds, say, k customers, in steady-state is:

$$P(N \geq k) = \rho^k \qquad (32)$$

We have adopted a single-channel queueing system (M/M/1) with Poisson arrivals and exponential service rate and arrivals are handled on a first come first serve basis, in this queueing system the average arrival rate is less than the average service rate (i.e. $\lambda < \mu$). In this case, there would be an unending queue.

The following formulas are developed for this system.

The average (or expected) number of customers in the queue at any time t is:

$$E(N_q) = L_q = \frac{\rho^2}{1-\rho} \qquad (33)$$

The average (or expected) number of customers waiting to be served at time t is:

$$\frac{1}{1-\rho} \qquad (34)$$

The average (or expected) number of customers in the system is:

$$E(N) = L = \frac{\rho}{1-\rho} \qquad (35)$$

The average (or expected) time a customer spends or wait in queue (before service is rendered), also called average waiting time, is:

$$E(W_q) = T_q = \frac{\rho}{\mu(1-\rho)} \qquad (36)$$

The average (or expected) time a customer spends or wait in the system (on the queue and receiving service, also called the average waiting time in the system, is: $\boldsymbol{E(W) = T = \frac{1}{\mu(1-\rho)}}$ **(37)**

V. Results

Data on arrivals, waiting times, service patterns and departures of customers in Intercontinental Bank PLC, Ile-Ife, Osun State, Nigeria were observed and collected for upwards of 21 working days between the working hours of 8.00 a.m. and 4.00 p.m. on daily basis. The waiting times and service times were obtained respectively by subtracting arrival times from the time service began; and subtracting the time service began from when it ended. On the final analysis, it was found that a total arrival rate of 1,302 customers per a total of 11,304 minutes (waiting times). Also a total service rate of 1,414 customers per a total service times of 10,284 minutes.

The following results are arrived at:

1. The arrival rate $\lambda = \frac{1{,}302}{11{,}304} = 0.1152$
2. The service rate $\mu = \frac{1{,}414}{10{,}284} = 0.1375$
3. The traffic intensity $\rho = \frac{\lambda}{\mu} = \frac{0.1152}{0.1375}$ $= 0.8378$
4. The average number of customers in the queue (i.e. number of customers waiting for service) $= \frac{\rho^2}{1-\rho} = \mathbf{4.327}, \boldsymbol{approx.\,4}$
   Thus there would be an average of 4 customers waiting for service.
5. The expected number of customers in the system (i.e. in the queue and the one being served) $= \frac{\rho}{1-\rho} = \mathbf{5.1652}$
6. The expected number of customers waiting to be served at any time t (when the queue exists) $= \frac{1}{1-\rho} = \mathbf{6.1652}$
7. The expected time a customer spends or waits in the queue before being served $\frac{\rho}{\mu(1-\rho)} = \mathbf{37.5653}$, $\boldsymbol{approx.\,38\ minutes}$
   Alternatively, $\frac{average\ no.of\ customers\ in\ the\ system}{\mu(i.e. service\ rate)} = \frac{5.1652}{0.1375} = 37.5650$ $(aprox, 38\ mins.$
8. The expected time a customer spends in the system (on queue and being served) $\frac{1}{\mu(1-\rho)} = 44.8380, approx.\,45\ mins.$
9. Average time of service $= average\ time\ in\ the\ system - average\ time\ in\ the\ queue$ $= 45mins. - 38mins = 7mins.$
10. The probability of queueing on arrival $= traffic\ intensity\ or\ utilization\ factor$ $= \rho = \frac{\lambda}{\mu} = 0.8378$ i.e. probability of at least one customer in the system.
11. The probability of no customer in the system or probability of not queueing on arrival is $1 - \frac{\lambda}{\mu}\ or\ (1-\rho)$
    $= 1 - 0.8378 = 0.1622$
12. The probability that there are n customers in the system (or the queue length is $P_n = \rho^n(1-\rho)$
    Hence, the probability of having a queue which is the probability of having two or more customers in the system is $1 - P_0 - P_1$ $(where\ P_0 = \rho^0(1-\rho)\ \&\ P_1 = \rho^1(1-\rho)$
    Now, $P_0 = 0.1622\ \&\ P_1 = 0.1359\ \therefore$
    $The\ prob.of\ having\ a\ queue\ is$
    $1 - 0.1622 - 0.1359 = 0.7019$

VI. Discussion of Results

The probability that a customer who arrives in the bank has to queue which is $1 - P_0$ which corresponds to traffic intensity (or the utilization factor) $\rho = 0.8378$ clearly indicates that there is always a very high possibility that customers would have to wait for every transaction in the bank since the bank officials concerned would always be busy attending to a customer that has earlier arrived. This is corroborated by $result\ 6$ which indicates that up to six customers will always be waiting to be attended to at any time t.It then follows that there will always be queue at any time t since the average time in the queue system (both on queue and receiving service) ($result\ 8$) is greater than the average time in the queue ($result\ 7$) before service is rendered.Also from $result\ 8$ which shows the average time that an individual customer would spend in the bank i.e. $45\ minutes$ before completing his/her transactions is on the higher side than expected in an ideal situation.

With all these results, we cannot say that the service in the bank is all that efficient. This is the situation in virtually most banks in the country. Therefore effort should always be made to reduce queue at least in our banks if it cannot be totally eliminated.

VII. Conclusion and Recommendation

Application of queueing theory is indeed a very useful and an indispensable statistical tool for solving problems of queueing in our banking sector. The queueing problems witnessed in Intercontinental Bank, PLC, Ile-Ife branch are very much peculiar to other banks across the country. This social phenomenon in our banking sector has caused a lot of negative effects to customers and the affected banks. This include excessive wasting of man hours, chaos, unnecessary congestion in banking halls which may lead to suffocation and contraction of communicable diseases (or other health complications), indirect reduction of customers which may in turn bring about less cash, profits, investments and shareholders to the affected banks.As a result of the above and other similar problems, it is recommended that more banking officials (cashiers, accountants, administrators, computer operators, analysts, etc.) should be employed so as to increase the channels of services.More ATM machines should be mounted in every nooks and crannies of cities, towns as well as villages as this will reduce the number of customers that will withdraw cash and check balances in the banks.Effective and efficient internet facilities which are reliable and fast should be installed for quick and easy access to individual statements of account during any bank transaction as this will reduce the waiting time of customers in banks. This should always be complemented by good sources of electricity e.g. electricity stand-by-generator which will at least bail one out of the epileptic power supply as always witnessed in the country at present.

In nutshell, more bank branches should be established along with the existing ones across the country, and adequately equipped with effective human/materials resources; current and effective banking management and operations should always be adopted for efficient and prompt banking services.

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